

# Region of attraction analysis of nonlinear stochastic systems using Polynomial Chaos Expansion

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## Abstract

A method is presented to estimate the region of attraction (ROA) of stochastic systems with finite second moment and uncertainty-dependent equilibria. The approach employs Polynomial Chaos (PC) expansions to represent the stochastic system by a higher-dimensional set of deterministic equations. We first show how the equilibrium point of the deterministic formulation provides the stochastic moments of an uncertainty-dependent equilibrium point of the stochastic system. A connection between the boundedness of the moments of the stochastic system and the Lyapunov stability of its PC expansion is then derived. Defining corresponding notions of a ROA for both system representations, we show how this connection can be leveraged to recover an estimate of the ROA of the stochastic system from the ROA of the PC expanded system. Two optimization programs, obtained from sum-of-squares programming techniques, are provided to compute inner estimates of the ROA. The first optimization program uses the Lyapunov stability arguments to return an estimate of the ROA of the PC expansion. Based on this result and user specifications on the moments for the initial conditions, the second one employs the shown connection to provide the corresponding ROA of the stochastic system. The method is demonstrated by two examples.

*Key words:* Region of Attraction, Stochastic Systems, Polynomial Chaos Expansion, Sum-of-Squares

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## 1 Introduction

The analysis of the region of attraction (ROA) of an uncertain nonlinear system is an active field of research (Chesi 2004, Valmorbida and Anderson 2017, Iannelli *et al.* 2019). The type of uncertainty and its appearance in the dynamical equations is often pivotal for the choice of the analytical approach. A class of uncertain systems commonly considered has two characteristic properties: firstly, the equilibrium point of the system is independent of the uncertainty, and secondly, the uncertainty comes from a uniform distribution. The stability of this class of systems can be analysed using Lyapunov methods where an estimate of the ROA is obtained in the form of the sublevel set of a Lyapunov function (Chesi *et al.* 2005). The aim then lies in finding a Lyapunov function verifying a largest possible estimate of the ROA. For systems where the uncertainty itself is parametric and polytope-bounded, parameter-dependent as well as common and composite Lyapunov functions have been investigated in, e.g., Topcu *et al.* (2010), Chesi (2004), Iannelli *et al.* (2019). While estimates for these cases can be efficiently obtained, the assumption of uncertainty-independent equilibria and uniformly distributed uncertainty excludes most systems from the analysis as equilibria are in general uncertainty-dependent and the stochasticity affecting the system can come from a wide range of distributions.

The ROA analysis in the case of uncertainty-dependent equilibria is not directly amenable to the use of Lyapunov functions, as this method requires knowledge of the equilibrium's location in the standard case. To

tackle this problem, an equilibrium-independent version of the ROA was proposed in Iannelli *et al.* (2018) where the idea is to formulate the ROA as a function of a new coordinate representing the deviation of the state relative to the equilibrium point. This approach, however, is still limited to uncertainties from uniform distributions. A more general approach for stability analysis is provided by contraction methods which inherently do not require knowledge on the equilibrium state. Contraction of uncertain systems was studied, e.g. in Ahbe *et al.* (2018) for polytope-bounded parametric uncertainty and in Pham *et al.* (2009) for Itô stochastic differential equations. Contraction methods often pose, however, numerically more complex problems compared to Lyapunov analysis as they consider the differential system. Furthermore, while contraction analysis gives conclusions about the contractive behaviour of a system it in general does not provide information on the state of the (stochastic) equilibrium.

In this work we present an efficient method to analyse the ROA of stochastic nonlinear systems with uncertainty-dependent equilibrium points where the uncertainty can be in form of any square-integrable random variable. The stochastic system is thereby represented by a higher-dimensional set of deterministic equations obtained from a Polynomial Chaos (PC) expansion of the stochastic dynamics. PC expansions are a polynomial approximation method which allow the representation of a second order random process, i.e. stochastic systems with finite second moment, by a higher-dimensional deterministic expression. An overview of PC expansions can be found, e.g., in Sullivan (2015) and Le Maitre

and Knio (2010). While PC expansion techniques have become established tools in uncertainty quantification, their use in stability and control is still sparse (Kim *et al.* 2013) and mostly focused on linear systems. Stability analysis of linear stochastic systems via PC expansions using Lyapunov inequalities was previously performed in Fisher and Bhattacharya (2009) and Lucia *et al.* (2017). In Hover and Triantafyllou (2006), the evolution of the stochastic modes resulting from the PC expansion was used to obtain information on the stability of a nonlinear system. A more generalized approach for polynomial systems using Lyapunov arguments is briefly presented in Fisher and Bhattacharya (2008), however the method proposed therein can only be used to certify global stability properties.

This paper proposes a novel method to analyse the ROA of stochastic nonlinear systems with uncertainty-dependent equilibria by leveraging the PC expansion framework. We first show how an equilibrium point of the deterministic expression given by the PC expansion corresponds to an uncertainty-dependent equilibrium point of the stochastic system. The latter can be represented as a set, referred to as the *equilibrium set*, for which statistical information is directly obtained from the expansion coefficients.

For both the stochastic system and its PC expansion notions of local stability are provided, consisting in boundedness of moments for the first and asymptotic stability in the sense of Lyapunov for the second. It is then demonstrated how Lyapunov stability of the PC equilibrium point implies moment boundedness of trajectories in the neighborhood of the equilibrium set of the stochastic system. From the stability notions and their shown connection, corresponding notions of the ROA are defined for both system representation. To obtain an inner estimate of the ROA of the PC expanded system, Lyapunov arguments stating sufficient conditions are formulated and converted into an algorithm. The algorithm employs well-established sum-of-squares verification techniques to test polynomial positivity (Parrilo 2000) which were previously used for analysing the ROA of polynomial systems in, e.g. Tan and Packard (2006), Topcu *et al.* (2010) and others. We then proceed by providing a notion of the ROA of the stochastic system which is formulated on the basis of the ROA of its deterministic PC expansion. While the ROA of a deterministic system is clearly defined, the definition of an attractive region of uncertain system can be of various types. For stochastic systems a definition of the ROA can be derived from the type of stochastic stability under consideration. For an overview of the different definitions of stochastic stability see, e.g., Khasminskii (2012). A widely used notion for the ROA of uncertain systems is that of a ‘robust’ ROA, which is the intersection of the ROA’s obtained for each realization of the uncertainty. As it thus relates to the worst case, this notion is suitable for uncertainties with uniform distributions but less so for other distributions where the worst case is not of practical interest or exploiting the statistical information available gives less conservative results. A probabilistic ROA of an uncertainty-independent equilibrium point was investigated for Ito-stochastic system via Lyapunov functions in Gudmundsson and Hafstein (2018). In Steinhardt and Tedrake (2012) ‘safe sets’ of a controlled system with quantified failure probabilities were considered and

computed with a supermartingale approach. We here provide an approach in which the ROA is obtained in terms of the region of initial conditions with specified moment properties for which trajectories almost surely converge to the equilibrium set of the stochastic system. The moment properties of the initial condition consist of, for example, a fixed variance in the initial state and can be specified by the user. The proposed method is demonstrated by two examples from the literature.

### 1.1 Notation

Let  $(\Theta, \mathcal{F}, \mu)$  be a probability space, where  $\Theta$  is a sample space,  $\mathcal{F}$  is a  $\sigma$ -algebra of the subsets in  $\Theta$  and  $\mu$  is a probability measure on  $(\Theta, \mathcal{F})$ . The Lebesgue space is denoted by  $\mathcal{L}_l$ , where  $1 \leq l \leq \infty$ . The inner product in the  $\mathcal{L}_2$  space is denoted by  $\langle \cdot, \cdot \rangle_{\mathcal{L}_2(\mu)}$  which represents integration (i.e. expectation, also indicated by  $\mathbb{E}$ ) with respect to  $\mu$ . A random variable  $\xi : \Theta \rightarrow \mathbb{R}$  with finite second moment,  $\xi \in \mathcal{L}_2(\Theta, \mu)$ , is referred to as the stochastic germ. For clarity of presentation we here consider one-dimensional stochastic germs. The extension to vector valued  $\xi$  with independent components is straightforward, see e.g. Sullivan (2015). Let the  $P$ -th moment of a random variable  $\xi$  be given by  $M_P(\xi) = \mathbb{E}[|\xi|^P]$ . A probability distribution  $\lambda$  with  $P$  given moments, where  $1 \leq P < \infty$ , is denoted by  $\lambda(M_{1..P})$ . The symbol  $\sim$  denotes an element with distribution  $\lambda$ .

Let  $\mathcal{P}^n$  denote the ring of all  $n$ -variate polynomials with real coefficients and let  $\mathcal{P}_{\leq r}^n$  denote those polynomials of total degree at most  $r \in \mathbb{N}_0$ . A polynomial  $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $g(x) \in \mathcal{P}_{\leq r}^n$ , is called a sum-of-squares (SOS) if it can be written as  $g(x) = \sum_i q_i(x)^2$ ,  $q_i(x) \in \mathcal{P}_{\leq r/2}^n$ . Moreover,  $g$  is SOS if and only if there is a matrix  $Q \succeq 0$  such that  $g(x) = v(x)^T Q v(x)$ , where  $v(x)$  is a vector of monomials. The set of all SOS polynomials in the indeterminate  $x$  is indicated by  $\Sigma[x]$ . The degree of a polynomial  $g$  in  $x$  is indicated by  $\partial(g)$ .

## 2 Problem Statement and Background

In this work we are interested in estimating the region of attraction of the equilibrium state of a stochastic nonlinear system.

The systems we consider are continuous time second order random processes of the form

$$\dot{x}(t, \xi) = f(x(t, \xi), a(\xi)), \quad (1)$$

where  $x(t, \xi) \in \mathbb{R}^n$  is the random state variable,  $a(\xi) \in \mathcal{L}_2(\Theta, \mu; \mathbb{R}^m)$  is a random variable, and  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is assumed to be polynomial in  $x$  and  $a$ . Further, we assume that  $\xi$  has finite support. In most practical applications this is the case and uncertainty distributions with typically infinite support, such as Gaussian distributions, can here be considered to have finite support with in practice negligible approximation error (Hover and Triantafyllou 2006).

We consider systems with an uncertainty-dependent attractive equilibrium point  $x_{EP}(\xi)$ . Let the set, given by the evaluation of  $x_{EP}(\xi)$  for each realization of the uncertainty, be denoted by

$$\mathcal{I} = \{x \in \mathbb{R}^n \mid f(x, a(\xi)) = 0, \xi \in \mathcal{L}_2(\Theta, \mu)\}. \quad (2)$$

In the following, the set  $\mathcal{I}$  of a system is referred to as the *equilibrium set*.

Let  $\psi(t, x_{\text{ini}}(\xi), \xi)$  denote the uncertainty-dependent solution of (1) at time  $t$  with initial condition  $x_{\text{ini}}(\xi)$ , where the initial state is also allowed to be random, i.e.  $x(t=0) = x_{\text{ini}}(\xi)$ . The ROA of the equilibrium set  $\mathcal{I}$  is then defined as

$$\mathcal{R}^* = \{x_{\text{ini}} \in \mathbb{R}^n \mid \mathbb{P}[\lim_{t \rightarrow \infty} d(\psi(t, x_{\text{ini}}(\xi), \xi), \mathcal{I}) = 0] = 1\}, \quad (3)$$

where  $\mathbb{P}$  denotes probability, and  $d$  is the distance measured in a chosen norm (e.g. the Euclidean norm).

### 2.1 Polynomial Chaos Expansion

Polynomial Chaos (PC) expansion can be used to approximate stochastic processes with finite second moment (which includes most stochastic processes of the physical world (Xiu and Karniadakis 2003)) by a higher dimensional set of deterministic equations. Most of the notations and definitions used in this section can be found e.g. in Sullivan (2015), Le Maître and Knio (2010). The PC expansion is performed within an orthogonal polynomial basis where the basis is chosen according to the type of probability distribution of the random variable in order to obtain optimal (in the  $\mathcal{L}_2$ -sense) convergence of the expansion. This is the case if the weighting function of the orthogonality relationship of the polynomial basis is identical to the probability function of a random distribution.

For a given probability space, an orthogonal polynomial basis is defined as follows.

**Definition 1** Let  $\mu$  be a non-negative measure on  $\Theta$ . A set of polynomials  $\mathcal{Q} = \{\Phi_i \mid i \in \mathbb{N}\} \subseteq \mathcal{P}$  is called an orthogonal system of polynomials if for each  $i \in \mathbb{N}$ ,  $\partial(\Phi_i) = i$ ,  $\Phi_i \in \mathcal{L}_2(\Theta, \mu)$  and

$$\langle \Phi_i(\xi), \Phi_j(\xi) \rangle = \int_{\Theta} \Phi_i(\xi) \Phi_j(\xi) d\mu(\xi) = \gamma_i \delta_{ij}, \quad (4)$$

where

$$\gamma_i := \langle \Phi_i(\xi), \Phi_i(\xi) \rangle \quad (5)$$

are (non-negative) normalization constants of the basis.

The orthogonal polynomial basis is constructed using a normalization such that  $\Phi_0 = 1$ . For any complete orthogonal basis of the Hilbert space  $\mathcal{L}_2(\Theta, \mu)$  the PC expansion is then defined as follows.

**Definition 2** Let  $y(\xi) \in \mathcal{L}_2(\Theta, \mu)$  be a square-integrable vector-valued random variable in  $\mathbb{R}^m$ ,  $m \in \mathbb{N}$ . The Polynomial Chaos expansion of  $y(\xi)$  with respect to the stochastic variable  $\xi$  is the expansion of  $y(\xi)$  in the orthogonal basis  $\{\Phi_i\}_{i=0}^p$

$$y(\xi) = \sum_{i=0}^p \bar{y}_i \Phi_i(\xi) \in \mathbb{R}^n, \quad (6)$$

with vector valued polynomial chaos coefficients  $\bar{y}_i \in \mathbb{R}^n$ ,  $\bar{y}_i = [\bar{y}_{i1}, \dots, \bar{y}_{in}]^T$ , which are obtained from

$$\bar{y}_i = \frac{\langle y(\xi), \Phi_i(\xi) \rangle}{\gamma_i}. \quad (7)$$

With  $p \rightarrow \infty$  the series in (6) becomes an exact expansion of  $y(\xi)$ .

The coefficients  $\{\bar{y}_i\}_{i \in \mathbb{N}_0}$  can be obtained by computing the integral in equation (7) for each component of  $y$  using, e.g., Galerkin projection.

### 2.2 Truncation error

For practical purposes, a PC expansion needs to be truncated for a specified order  $p$ . As the expansion series is  $\mathcal{L}_2$ -convergent for second order random processes, low orders of  $p$  are in general sufficient to keep the error introduced by the truncation small and represent the original system sufficiently well (Sullivan 2015, Xiu and Karniadakis 2002). Thus, in the remainder of the paper the following working assumption is made.

**Assumption 1** The PC expanded system (15) truncated at order  $p$  accurately represents the stability properties of the true stochastic system (1).

An analysis of possible effects of the truncation order can be found in Field and Grigoriu (2004). In case a guaranteed accuracy of the truncated system is required the truncation error can be upper bounded and added to the expansion as model uncertainty, see, e.g., Mühlpfordt et al. (2018), and Fagiano et al. (2011).

### 2.3 PC expansion of moments

In the PC framework the moments of a random variable or stochastic process can be retrieved from the coefficients of the  $\mathcal{L}_2$ -optimal expansion. Let  $y(\xi) \in \mathbb{R}^n$  be a random variable. With the notation in (6), the  $P$ -th moment, where  $1 \leq P < \infty$ , is obtained from

$$\begin{aligned} \mathbb{E}[|y(\xi)|^P] &= \sum_{i,j,\dots,P=0}^p \bar{y}_i \bar{y}_j \dots \bar{y}_P \langle \Phi_i \Phi_j \dots, \Phi_P \rangle \\ &=: \bar{M}_{1..P}(\bar{y}). \end{aligned} \quad (8)$$

For the first moment, i.e. the mean, (8) results in

$$m(y(\xi)) := \mathbb{E}[y(\xi)] = \langle y(\xi), \Phi_0 \rangle = \bar{y}_0. \quad (9)$$

Further, for the variance of  $y(\xi)$ , we obtain

$$\mathbb{E}[|y(\xi) - \mathbb{E}[y(\xi)]|^2] = \sum_{j=1}^p \bar{y}_j^2 \gamma_j, \quad (10)$$

and for the covariance matrix  $\sigma$  of  $xy(\xi)$

$$\sigma(y(\xi)) := \sum_{j=1}^p \bar{y}_j \bar{y}_j^T \gamma_j, \quad (11)$$

where, in particular, we have for each entry of the matrix  $\sigma_{kl} = \sum_{j=1}^p \bar{y}_{k_j} \bar{y}_{l_j} \gamma_j$ , with  $\bar{y}_{k_j}, \bar{y}_{l_j}$  representing the  $j$ -th PC coefficients of the  $k$ -th, respectively  $l$ -th, component of the random variable  $y \in \mathbb{R}^n$ .

In the remainder of the paper we will denote any PC expansion coefficient or variable dependent on such with an overbar-notation to distinguish them from the stochastic variables. Since the mean and the variance are given by separate sets of PC coefficients, the following notation is used for the coefficients of the PC expansion of  $y \in \mathbb{R}^m$ .

$$\bar{y}_0 := [\bar{y}_{1_0}, \dots, \bar{y}_{m_0}]^T \in \mathbb{R}^m, \quad (12)$$

$$\bar{y}_J := [\bar{y}_{1_1}, \dots, \bar{y}_{m_1}, \dots, \bar{y}_{1_p}, \dots, \bar{y}_{m_p}]^T \in \mathbb{R}^{m \cdot p}, \quad (13)$$

$$\bar{y} := [\bar{y}_0, \bar{y}_J]^T \in \mathbb{R}^{m \cdot (p+1)}, \quad (14)$$

where the elements in  $\bar{y}_0$  are called the *mean modes*, the elements in  $\bar{y}_J$  the *variance modes* and together they are referred to as *stochastic modes*  $\bar{y}$ .

#### 2.4 PC expansion of stochastic polynomial ODEs

Applying the PC expansion to *stochastic* dynamical systems results in a *deterministic* representation of the system at the expense of an increased state dimension. More precisely, by expanding the random variables up to order  $p$  and projecting the resulting expansion onto each of the  $p$  basis functions, the  $n$ -dimensional stochastic system is represented by a  $n \cdot (p+1)$ -dimensional deterministic system. We use the notation

$$\dot{\bar{x}} := \bar{f}(\bar{x}), \quad (15)$$

where  $\bar{x} \in \mathbb{R}^{n(p+1)}$  is the vector of PC expansion coefficients, and  $\bar{f} : \mathbb{R}^{n(p+1)} \rightarrow \mathbb{R}^{n(p+1)}$ , to refer to the dynamics resulting from the PC expansion of a stochastic system (1). The expansion is demonstrated for an example system where  $n = 1$ .

$$\dot{x}(t, \xi) = a(\xi)x^3(t, \xi). \quad (16)$$

Expanding (16) and dropping the  $(\xi)$  and  $(t)$ -notation for clarity results in

$$\sum_{i=0}^p \dot{\bar{x}}_i \Phi_i = \sum_{j,k,l,m=0}^p \bar{a}_j \bar{x}_k \bar{x}_l \bar{x}_m \Phi_j \Phi_k \Phi_l \Phi_m. \quad (17)$$

Projecting (17) onto the  $q$ -th basis polynomial

$$\sum_{i=0}^p \dot{\bar{x}}_i \langle \Phi_i, \Phi_q \rangle = \sum_{j,k,l,m=0}^p \bar{a}_j \bar{x}_k \bar{x}_l \bar{x}_m \langle \Phi_j \Phi_k \Phi_l \Phi_m, \Phi_q \rangle, \quad (18)$$

we obtain  $q$  deterministic differential equations

$$\dot{\bar{x}}_q = \gamma_q^{-1} \cdot \sum_{j,k,l,m=0}^p \bar{a}_j \bar{x}_k \bar{x}_l \bar{x}_m \langle \Phi_j \Phi_k \Phi_l \Phi_m, \Phi_q \rangle. \quad (19)$$

**Remark 1** *The polynomial basis for the PC expansion of (16) is often chosen according to the  $\mathcal{L}_2$ -optimal basis for  $\xi$ . While the PC expansion for a second order random process such as (16) can be performed in any basis as given by Definition 1, the convergence of the expansion will be faster or slower depending on the choice. This translates into the truncation order  $p$  needed to represent the system sufficiently accurately by the expansion, with slower convergence implying larger  $p$ .*

### 3 Stability of Stochastic Systems

We are interested in analysing the stability properties of the equilibrium set of a stochastic system (1) by means of its PC expansion (15). In order to draw conclusions from the stability properties of the PC expansion on the stability of the stochastic system, a connection between the behavior of both systems is established.

#### 3.1 Relationship of equilibria

Before stating the notions of stability we first show the relationship between the equilibria of (1) and (15).

**Lemma 1** *The stochastic system (1) has an equilibrium set  $\mathcal{I}$  as defined in (2) if and only if the PC expanded system has an equilibrium point,  $\bar{x}_{EP} \in \mathbb{R}^{n(p+1)}$ .*

**Proof.** Let  $f(x_{EP}(\xi)) = 0$ . The PC expansion of  $f(x_{EP}(\xi))$  is  $f(\bar{x}_{EP})$ , where  $x_{EP}(\xi) = \sum_{i=0}^p \bar{x}_{EPi} \Phi_i(\xi)$  from (6). Assume  $\bar{x}_{EP}$  was not an equilibrium of  $\bar{f}$ , i.e.  $f(\bar{x}_{EP}) \neq 0$ . Then there exists a  $t > 0$ ,  $\bar{\psi}(t, \bar{x}_{EP}) = \bar{x}(t) \neq \bar{x}_{EP}$ . However,  $\bar{\psi}(t, \bar{x}_{EP})$  is the PC expansion of  $\psi(t, x_{EP})$ , and, by equation (2),  $\psi(t, x_{EP}(\xi)) = \psi(0, x_{EP}(\xi)) = x_{EP}(\xi)$ , so  $\bar{\psi}(t, \bar{x}_{EP}) = \bar{x}_{EP}$ . This argument holds both ways, and thus  $f(x_{EP}(\xi)) = 0 \Leftrightarrow \bar{f}(\bar{x}_{EP}) = 0$ .  $\square$

The equilibrium set can be obtained numerically by explicit computation of the expansion  $x_{EP}(\xi) = \sum_{i=0}^p \bar{x}_{EPi} \Phi_i(\xi)$ . Using equation (8) for a known  $\mathcal{L}_2$ -optimal basis, the equilibrium set can further be expressed in terms of its moments,  $\mathcal{I} = \{x \in \mathbb{R}^n \mid x \in x_{EP}(\xi) \sim \lambda(\bar{M}_{1..P}(\bar{x}_{EP}))\}$ .

Due to Lemma 1 the task of analysing the stability of the uncertainty-dependent equilibrium point of the stochastic system converts to the well-known problem of analysing the stability of an equilibrium point of a deterministic system. Moreover, it emphasizes the important aspect that an equilibrium point of the PC expanded system not only corresponds to an equilibrium set of the stochastic system but also contains the statistical information of the set. Note that the location of  $\bar{x}_{EP}$  can be easily obtained by simulating a trajectory of (15) with initial state in the region of interest.

**Remark 2** *If the variance modes of  $\bar{x}_{EP}$  are zero, i.e.  $\bar{x}_{EPJ} = 0$ , then the stochastic system has an uncertainty-independent equilibrium point located at  $x_{EP} = \bar{x}_{EP0}$ . The equilibrium set  $\mathcal{I}$  thus only contains one element. Moreover, if all stochastic modes are zero,  $\bar{x}_{EP} = 0$ , then also  $x_{EP} = 0$ .*

Based on this relationship between the equilibria we propose a connection between certain stability notions which are specified for each system in the following.

#### 3.2 $P$ -th moment boundedness and stability

For stochastic systems there are various concepts of stability ranging from weaker forms such as stability in probability to stronger forms such as  $P$ -th moment stability up to almost sure stability, see, e.g. Kozin (1969) for an overview. In the following we focus on  $P$ -th moment boundedness and stability of stochastic systems

where we employ the definitions as found in, e.g., Khasminskii (2012), Wu and Meng (2004), Khalil (2002):

**Definition 3** *The solutions of (1) are called ultimately bounded in the  $P$ -th moment if there exists a  $c > 0$  such that for any  $b > 0$  there exists a  $T = T(b) > 0$  such that*

$$|x_{ini}| < b \rightarrow \mathbb{E}[|x(t, \xi)|^P] < c, \quad \forall t \geq T. \quad (20)$$

*Further, if there is only one element in  $\mathcal{I}$  then let this element, without loss of generality, be the zero point. This zero point is called stable in the  $P$ -th moment, if for each  $\epsilon > 0$ , there exists a  $\delta > 0$  such that*

$$|x_{ini}| < \delta \rightarrow \mathbb{E}[|x(t, \xi)|^P] < \epsilon, \quad \forall t \geq 0, \quad (21)$$

*and asymptotically stable in the  $P$ -th moment if, further,*

$$|x_{ini}| < \delta \rightarrow \mathbb{E}[|x(t, \xi)|^P] \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (22)$$

We now define a suitable notion of stability for the PC expanded system. As we are interested in equilibrium points of the PC expansion and, further, the PC expanded system is deterministic, we use stability in the sense of Lyapunov.

**Definition 4** *The equilibrium point  $\bar{x}_{EP}$  of (15) is locally stable if for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that*

$$|\bar{x}_{ini}| < \delta \Rightarrow |\bar{x}(t) - \bar{x}_{EP}| < \epsilon, \quad \forall t > 0. \quad (23)$$

*Further,  $\bar{x}_{EP}$  is locally asymptotically stable if it is locally stable and  $\delta$  can be chosen such that*

$$|\bar{x}_{ini}| < \delta \Rightarrow |\bar{x}(t) - \bar{x}_{EP}| \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (24)$$

With Definition 4 we find the following result for the stochastic system.

**Theorem 1** *Let the system (15) with  $\bar{f} : \bar{D} \rightarrow \bar{D} \subseteq \mathbb{R}^{n \cdot (p+1)}$  be the PC expansion of the stochastic system (1). If the equilibrium point  $\bar{x}_{EP} \in \bar{D}$  is locally asymptotically stable then the solutions of the stochastic system (1) are ultimately bounded in the  $P$ -th moment in a neighborhood of  $\mathcal{I}$ . If, further,  $\bar{x}_{EP}$  represents a  $\mathcal{I}$  containing a single point, then (1) is locally asymptotically stable in the  $P$ -th moment.*

**Proof.** If  $\bar{x}_{EP}$  is an equilibrium point of (15) then every trajectory  $\bar{x}(t)$  in a neighborhood of  $\bar{x}_{EP}$  will eventually converge to  $\bar{x}_{EP}$ . As all components  $\bar{x}_i(t)$  in this case converge to a finite value, so does every term in the expression in (8) and thus  $\mathbb{E}[|x(t, \xi)|^P]$  will eventually converge to a finite value, which is given by inserting  $\bar{x}_{EP}$  into the right hand side of equation (8). The ultimate boundedness of the  $P$ -th moment as defined in (20) follows. If the equilibrium point  $\bar{x}_{EP}$  represents an  $\mathcal{I}$  consisting of a single point then this implies that  $\bar{x}_{EP, J} = 0$  (see Remark 2). Thus, every component of  $\bar{x}_J(t)$  will converge to zero and every component of  $\bar{x}_0(t)$  will converge to  $\bar{x}_{EP0}$  as  $t \rightarrow \infty$ . Assuming without loss of generality  $\bar{x}_{EP0} = 0$ , it follows that equation (8) converges to zero and thus equation (22) holds.  $\square$

**Remark 3** *Note that the reverse is not true: ultimately bounded solutions of the stochastic system (1) do not imply a convergence of the components  $\bar{x}(t)$  to constant values. One example for this is readily provided by systems with a stable limit cycle. The trajectories in a neighborhood of the limit cycle converge to the limit cycle and thus are locally ultimately bounded, however the PC expansion coefficients  $\bar{x}_i(t)$  do not converge to an equilibrium point but instead remain ultimately bounded to a set as well.*

Theorem 1 allows us to obtain information about the behavior of the stochastic system by analysing the local stability properties of an equilibrium point  $\bar{x}_{EP}$  of the PC expanded system. In the following we formulate the criteria with which the attractive region of  $\bar{x}_{EP}$  can be obtained.

#### 4 PC Expansion-based Region of Attraction Analysis

In this section we first define the ROA of an equilibrium point  $\bar{x}_{EP}$  of the PC expanded system and state the criteria with which an inner estimate of it can be obtained. We then show how this ROA translates to an inner estimate of  $\mathcal{R}^*$ , the ROA of the stochastic system. Finally, optimization programs to maximize inner estimates of both ROAs are proposed.

##### 4.1 Formulation of the ROA based on a PC Expansion

Let the ROA of  $\bar{x}_{EP}$  be defined by the set

$$\bar{\mathcal{R}}^* = \{\bar{x}_{ini} \in \mathbb{R}^{n \cdot (p+1)} \mid \lim_{t \rightarrow \infty} d(\bar{\psi}(t, \bar{x}_{ini}), \bar{x}_{EP}) = 0\}, \quad (25)$$

where  $\bar{\psi}(t, \bar{x}_{ini})$  denotes the solution of the PC expanded system at time  $t$  with initial state  $\bar{x}_{ini}$ . An inner estimate of  $\bar{\mathcal{R}}^*$ , denoted by  $\bar{\mathcal{R}}$ , is then obtained from the following arguments.

**Theorem 2** *Let  $\bar{D} \subset \mathbb{R}^{n \cdot (p+1)}$  be a compact domain containing  $\bar{x}_{EP}$  and let  $V$  be a continuously differentiable function  $V(\bar{x}) : \bar{D} \rightarrow \mathbb{R}$ . For a scalar  $\rho > 0$  let  $\Omega_{V_\rho} = \{\bar{x} \in \bar{D} \mid V(\bar{x}) \leq \rho\}$  be the  $\rho$ -sublevel set of  $V$ . If  $V$  satisfies*

$$V(\bar{x}) > 0 \quad \forall \bar{x} \in \Omega_{V_\rho} \setminus \{\bar{x}_{EP}\}, \quad V(\bar{x}_{EP}) = 0, \quad (26)$$

$$\dot{V}(\bar{x}) < 0 \quad \forall \bar{x} \in \Omega_{V_\rho} \setminus \{\bar{x}_{EP}\}, \quad (27)$$

*then  $V$  is a Lyapunov function and every trajectory  $\bar{x}_{ini}$  starting in  $\Omega_{V_\rho}$  will converge to  $\bar{x}_{EP}$  as  $t \rightarrow \infty$ . Thus, the set  $\bar{\mathcal{R}} = \{\bar{x}_{ini} \in \bar{D} \mid \bar{x}_{ini} = \bar{x}, \forall \bar{x} \in \Omega_{V_\rho}\}$  is an inner estimate of  $\bar{\mathcal{R}}^*$ .*

The proof uses Lyapunov arguments which are standard in ROA analysis and can be found, e.g. in Khalil (2002). Theorem 2 presents a criterion for a set  $\bar{\mathcal{R}}$  to be an estimate of the ROA, where  $\bar{\mathcal{R}}$  is in terms of the PC expansion coefficients. We now provide the means to infer information about  $\bar{\mathcal{R}}^*$ , the ROA of the equilibrium set  $\mathcal{I}$  of the stochastic system, from  $\bar{\mathcal{R}}$ . More precisely, we show how the inner estimate  $\bar{\mathcal{R}}$  translates into an inner estimate  $\mathcal{R}$  of the stochastic ROA. Recalling the expression (3) for the ROA of the equilibrium set  $\mathcal{I}$  of a stochastic system, the following arguments can be made.

**Lemma 2** Let  $\bar{\mathcal{R}}$  be an inner estimate of the ROA of  $\bar{x}_{EP}$ ,  $\bar{\mathcal{R}} \subseteq \bar{\mathcal{R}}^*$ . Then the set

$$\mathcal{R} = \{x_{ini} \in \mathbb{R}^n \mid x_{ini}(\xi) \sim \lambda(\bar{M}_{1..P}(\bar{x}_{ini})), \forall \bar{x}_{ini} \in \bar{\mathcal{R}}\}, \quad (28)$$

is a subset of the ROA of  $x_{EP}$ ,  $\mathcal{R} \subseteq \mathcal{R}^*$ .

**Proof.** We first establish the relationship between  $x_{ini}(\xi)$  and  $\bar{x}_{ini} \in \bar{\mathcal{R}}$ . The PC coefficients  $\bar{x}_{ini} \in \bar{\mathcal{R}}$  represent the stochastic variables  $x_{ini}(\xi)$  by the relation (6), such that any  $x_{ini}^\#(\xi) \in \mathcal{R}$  is given by  $x_{ini}^\#(\xi) = \sum_{i=0}^P \bar{x}_{ini}^\# \Phi_i(\xi)$ . For this  $x_{ini}^\#(\xi)$ , from equation (8) the moments are given by  $M_{1..P}(x_{ini}^\#) = \bar{M}_{1..P}(\bar{x}_{ini}^\#)$ . This reasoning holds for all  $x_{ini} \in \mathcal{R}$ . We now turn to prove  $\mathcal{R} \subseteq \mathcal{R}^*$ . Recall, that from Theorem 2 we have  $\bar{x}_{ini} \in \bar{\mathcal{R}} \implies \lim_{t \rightarrow \infty} \bar{\psi}(t, \bar{x}_{ini}) = \bar{x}_{EP}$ . Let further  $\bar{x}(t) = \bar{\psi}(t, \bar{x}_{ini})$  and  $x(t, \xi) = \psi(t, x_{ini}(\xi), \xi)$ . With (8) and from Theorem 1, it follows that if  $\bar{x}_{ini} \in \bar{\mathcal{R}}$  then

$$\begin{aligned} \mathbb{E}[|x(t, \xi)|^P] &= \sum_{i, \dots, P=0}^P \bar{x}_i(t) \cdots \bar{x}_P(t) \langle \Phi_i \cdots, \Phi_P \rangle, \\ \xrightarrow{t \rightarrow \infty} \sum_{i, \dots, P=0}^P \bar{x}_{EP_i} \cdots \bar{x}_{EP_P} \langle \Phi_i \cdots, \Phi_P \rangle &= \mathbb{E}[|x_{EP}(\xi)|^P], \quad (29) \end{aligned}$$

where  $1 \leq P < \infty$  and for a given  $x_{EP}(\xi)$  and  $P$  the term  $\mathbb{E}[|x_{EP}(\xi)|^P]$  is a constant. So far, we have shown the moment convergence of a random variable  $x_{ini}(t, \xi) \in \mathcal{R}$ . It remains to show that from this follows  $\lim_{t \rightarrow \infty} \mathbb{P}[d(\psi(t, x_{ini}(\xi), \xi), \mathcal{I}) = 0] = 1$  almost surely. To this end, recall that  $\xi$  has finite support and thus  $\Theta$  is bounded. Assume there is a subset  $\Theta^\dagger \subset \Theta$  for which  $\xi^\dagger \in \Theta^\dagger : d(x(t, \xi^\dagger), \mathcal{I}) \not\rightarrow 0$  as  $t \rightarrow \infty$ . Consider first the case where  $x(t, \xi^\dagger) \rightarrow \infty$  as  $t \rightarrow \infty$ . Then

$$\begin{aligned} \mathbb{E}[|x(t, \xi)|^P] &= \int_{\Theta} |x(t, \xi)|^P d\mu(\xi) \\ &= \int_{\Theta^\dagger} |x(t, \xi^\dagger)|^P d\mu(\xi^\dagger) + \int_{\Theta^{\dagger C}} |x(t, \xi^{\dagger C})|^P d\mu(\xi^{\dagger C}), \quad (30) \end{aligned}$$

where  $\xi^{\dagger C} \in \Theta^{\dagger C}$  and  $\Theta^{\dagger C}$  denotes the complement of  $\Theta^\dagger$ , such that  $\Theta^{\dagger C} \cup \Theta^\dagger = \Theta$ . The first term in (30) and by that the  $P$ -th moment of  $x(t, \xi)$  will, however, tend to infinity as  $t$  goes to infinity, unless the elements in  $\Theta^\dagger$  have  $\mu$ -measure zero. Consider now the case where  $d(x(t, \xi^\dagger), \mathcal{I}) \rightarrow c$  as  $t \rightarrow \infty$ , where  $0 < c < \infty$  is a constant. In order to not contradict (29), considering (30) we find that either  $x(t, \xi^\dagger) = x(t, \xi)$  for all  $\xi^\dagger = \xi$ , but this implies  $d(x(t, \xi^\dagger), \mathcal{I}) \rightarrow 0$  as  $t \rightarrow \infty$ , or  $\mu(\xi^\dagger) = 0$ . Hence, from moment convergence follows here the almost sure convergence of  $x(t, \xi)$  to  $\mathcal{I}$ , such that  $\lim_{t \rightarrow \infty} \mathbb{P}[d(\psi(t, x_{ini}(\xi), \xi), \mathcal{I}) = 0] = 1$  for all  $x_{ini} \in \mathcal{R}$  and thus  $\mathcal{R} \subseteq \mathcal{R}^*$ .  $\square$

If  $\xi$  has infinite support then almost sure convergence of trajectories from moment convergence cannot be concluded. Based on the proof above, the meaning of the computed region  $\mathcal{R}^*$  would change and could now be characterized as the region for which the *moments* of all trajectories starting in  $\mathcal{R}^*$  converge to the moments of  $\mathcal{I}$ .

## 4.2 Algorithmic computation of $\bar{\mathcal{R}}$

In the following we present algorithms by which  $\bar{\mathcal{R}}$  can be computed. In order to make the following implementations generalizable, a coordinate shift is introduced, similar to the one proposed in Iannelli *et al.* (2018):

$$\bar{z} = \bar{x} - \bar{x}_{EP}, \quad (31)$$

This shift centers the analysed system around the zero point. Note that while in Iannelli *et al.* (2018)  $\bar{x}_{EP}$  is not known because it depends on the uncertainty, in this formulation  $\bar{x}_{EP}$  is deterministic and can be obtained, e.g. by simulation of the PC expanded system.

Using polynomial functions for  $V$ , the conditions on the set  $\bar{\mathcal{R}}$  as stated in Theorem 2 are in polynomial form. This allows to employ an approach introduced in Parrilo (2000), and formulate the ROA conditions as semi-algebraic set emptiness conditions. These can be efficiently solved through a relaxation to sum-of-squares (SOS) programs employing Stengle's Positivstellensatz (Stengle 1974). Details on the procedure of formulating conditions such as those in Theorem 2 and Lemma 1 into SOS constraints are omitted for brevity and can be found in, e.g. Parrilo (2000), Tan and Packard (2006), and Topcu *et al.* (2010). The resulting SOS program consists of polynomial objectives and constraints. Each of the constraints is a requirement that the polynomial is SOS. Since an SOS constraint is a positive-definiteness constraint (see Section 1.1), the aim is to solve the SOS program as a semidefinite program (SDP).

Applying the procedure to the conditions on  $\bar{\mathcal{R}}$  as stated in Theorem 2 results in the following SOS program.

$$\max_{V(\bar{z}), s_1(\bar{z}), \rho} \text{vol}(\bar{\mathcal{R}}(\bar{z})) \quad (32a)$$

$$\text{subject to} \quad V(\bar{z}) - l(\bar{z}) \in \Sigma[\bar{z}], \quad (32b)$$

$$-\dot{V}(\bar{z}) - s_1(\bar{z})(\rho - V(\bar{z})) - l(\bar{z}) \in \Sigma[\bar{z}], \quad (32c)$$

$$s_1(\bar{z}) \in \Sigma[\bar{z}], \quad (32d)$$

where the multiplier  $s_1$  is an SOS polynomial of potentially arbitrarily high degree which results directly from the Positivstellensatz and, once obtained, certifies that the solution of the program adheres to the constraints. The term  $l(\bar{z})$  is an even polynomial with small fixed coefficients (e.g.,  $l(\bar{z}) = 10^{-4} \bar{z}^T \bar{z}$ ), which results from the definiteness of the conditions in (26) and (27) for all  $\bar{x}$  except for  $\bar{x}_{EP}$ .

In order for the problem (32) to be solvable as an SDP it has to be convex in the decision variables. This can be achieved by the following steps. The set  $\Omega_{V, \rho}$  is formulated as the sublevel set  $\Omega_{V, \rho=1} = \{\bar{z} \mid V(\bar{z}) := v(\bar{z})^T Q_V v(\bar{z}) \leq 1, Q_V > 0\}$  where  $v(\bar{z})$  is the vector of monomials in  $\bar{z}$  and  $\rho$  is fixed to 1 as optimizing over  $\rho$  is redundant when optimizing over  $Q_V$ . Furthermore, the objective in (32a) is a generic expression for the volume of the ROA and needs to be replaced by a convex expression. It has been previously observed (Tan and Packard 2006) that higher degree functions  $V$  have the potential to verify larger estimates of the ROA. For  $\partial(V) > 2$  the volume of a sublevel set cannot be computed from a convex expression and thus a surrogate that is a computationally tractable measure for the ROA is employed. We use a convex measure in the

form of the geometric mean of the eigenvalues of the matrix  $B$  of the sublevel set  $\mathcal{B} = \{\bar{z} \mid b := \bar{z}^T B \bar{z} \leq 1\}$  of a quadratic function  $b(\bar{z})$ . This geometric mean is a monotone function of the determinant, which itself is inversely proportional to the volume of the set. Minimizing this geometric mean thus maximizes the volume of a quadratic set. With the constraint that the surrogate set  $\mathcal{B}$  lies inside the sublevel set  $\Omega_{V_1}$ ,  $\mathcal{B} \subseteq \Omega_{V_1}$ , a maximization of the set  $\mathcal{B}$  leads to the estimate of  $\bar{\mathcal{R}}$  being increased simultaneously. Utilization of this surrogate set requires adding the following constraints to the optimization program (32):

$$-s_2(\bar{z})(1 - b(\bar{z})) - (1 - V(\bar{z})) \in \Sigma[\bar{z}], \quad (33a)$$

$$s_2(\bar{z}) \in \Sigma[\bar{z}]. \quad (33b)$$

The objective function (32a) is then replaced by the geometric mean of the eigenvalues of  $B$ ,

$$\min_{V, s_1, s_2, B} \det(B)^{1/n(p+1)}. \quad (34)$$

The resulting optimization program then consists in

$$\text{solve} \quad (34) \quad (35a)$$

$$\text{subject to} \quad (32b), (32c), (32d), (33a), (33b). \quad (35b)$$

This SOS program is bilinear in the multipliers  $s_1$ , respectively  $s_2$  and  $V$ , respectively  $B$ , which prevents its direct solution as an SDP. However, the optimal solution can be approximated by iteratively solving (35) as an SDP by fixing one of the two bilinear variables and optimizing over the other, and vice versa, until a pre-defined convergence tolerance is reached. This requires an initial estimate for  $\Omega_{V_1}$  which is obtained by solving the Lyapunov matrix inequality for the linearized state matrix at the equilibrium point, and scaling it suitably. Similarly, the initial estimate of the matrix  $B$  is found by using a suitably scaled unit diagonal matrix.

### 4.3 Recovering $\mathcal{R}$ from $\bar{\mathcal{R}}$

We propose an approach in form of an optimization problem in which the set  $\mathcal{R}$ , as given by Lemma 2 for initial conditions with specified stochastic properties, can be recovered from the set  $\bar{\mathcal{R}}$ . In particular, the program shows how to obtain a maximized estimate  $\mathcal{R}$  of the true ROA  $\mathcal{R}^*$  from a given set  $\bar{\mathcal{R}}$ . The set  $\mathcal{R}$  is given by *stochastic* variables  $x$  which represent the initial state, whose statistical properties are given by all possible states of the PC coefficients contained in  $\bar{\mathcal{R}}$ . In the set  $\bar{\mathcal{R}}$ , the mean modes  $\bar{x}_0$  and the variance modes  $\bar{x}_J$  can be traded off, allowing for a wide range of statistical properties. In order to obtain a set  $\mathcal{R}$  of the stochastic system in the  $x$  variables, one of the two statistical properties, either the mean or the covariance, of the initial states can be fixed and the set  $\mathcal{R}$  obtained in terms of the other. We here choose to fix the covariance of the initial states  $x$  to a specified level, which is denoted by  $\hat{\sigma}$ , and compute  $\mathcal{R}$  in terms of the mean of  $x$ . The  $\mathcal{R}$  obtained in this way will be denoted by  $\mathcal{R}_0$  in the following. Since  $m(x) = \bar{x}_0$  (equation (9)), the set  $\mathcal{R}_0$  is given by

$$\mathcal{R}_0 = \{\bar{x}_0 \in \bar{\mathcal{R}} \mid \bar{x} \in \bar{\mathcal{R}}, \sum_{j=1}^p \bar{x}_j \bar{x}_j^T \gamma_j = \hat{\sigma}\}. \quad (36)$$

The set  $\mathcal{R}_0$  can be computed from a given  $\bar{\mathcal{R}}$  by the following optimization problem. Let  $\mathcal{R}_0$  hereby be represented by the 1-sublevel set of the polynomial function  $\mathcal{R}_0 := \{\bar{x}_0 \mid v(\bar{x}_0)^T Q_0 v(\bar{x}_0) \leq 1\}$ . The aim is to maximize  $\mathcal{R}_0$  inside  $\bar{\mathcal{R}}$  while keeping the size of the polynomials in (11), representing the covariance of the initial states  $x_{\text{ini}}$ , fixed.

$$\max_{Q_0} \quad \text{vol}(\mathcal{R}_0) \quad (37a)$$

$$\text{subject to} \quad v(\bar{x})^T Q_V v(\bar{x}) \leq 1, \quad (37b)$$

$$\sum_{j=1}^p \bar{x}_j \bar{x}_j^T \gamma_j = \hat{\sigma}, \quad (37c)$$

$$v(\bar{x}_0)^T Q_0 v(\bar{x}_0) \leq 1, \quad (37d)$$

$$Q_0 > 0, \quad (37e)$$

$$\mathcal{R}_0 \subseteq \bar{\mathcal{R}}, \quad (37f)$$

where  $Q_V$  is the optimizer of (32). Note that (37c) is a matrix equality constraint with polynomial entries. Since  $\hat{\sigma}$  is a symmetric matrix, equation (37c) results in  $\frac{n(n+1)}{2}$  scalar constraints. As  $\partial(\mathcal{R}_0) = \partial(V)$ , a convex surrogate set similar to that in (33) is introduced to tractably maximize  $\mathcal{R}_0$  for  $\partial(\mathcal{R}_0) > 2$ . To this end we use a quadratic sublevel set in terms of the mean modes,  $\mathcal{B}_0 = \{\bar{x} \mid b_0 := \bar{x}_0^T B_0 \bar{x}_0 \leq 1\}$ , constrained to remain within  $\mathcal{R}_0$ . The following constraints are added to program (37) to give a convex optimization of a lower bound on the volume of  $\mathcal{R}_0$ .

$$b_0 \leq 1, \quad B_0 > 0, \quad \mathcal{B}_0 \subseteq \mathcal{R}_0. \quad (38)$$

The following optimization program shows the implementation of the problem in (37)-(38) that efficiently obtains an estimate of  $\mathcal{R}_0$ .

$$\min_{s_1, s_2, h_{lk}, \dots, Q_0, B_0} \det(B_0)^{1/n} \quad (39a)$$

subject to:

$$\begin{aligned} & -s_1(\bar{x})(1 - v(\bar{x}_0)^T Q_0 v(\bar{x}_0)) + (1 - v(\bar{x})^T Q_V v(\bar{x})) + \\ & + \sum_{l=1, k \geq l}^n h_{lk}(\bar{x})(\hat{\sigma}_{lk} - \bar{x}_{lJ}^T \Gamma \bar{x}_{kJ}) \in \Sigma[\bar{x}], \end{aligned} \quad (39b)$$

$$-(1 - b_0)s_2(\bar{x}_0) + (1 - v(\bar{x}_0)^T Q_0 v(\bar{x}_0)) \in \Sigma[\bar{x}], \quad (39c)$$

$$s_1(\bar{x}) \in \Sigma[\bar{x}], \quad (39d)$$

$$s_2(\bar{x}_0) \in \Sigma[\bar{x}_0]. \quad (39e)$$

The objective function is now the volume of the surrogate set  $\mathcal{B}_0$  represented by the geometric mean of the eigenvalues of the matrix  $B_0$ . The vector  $\bar{x}_{dJ} := [\bar{x}_{d1}, \dots, \bar{x}_{dp}]^T$  contains the variance modes of the  $d$ -th dimension with  $d = 1, \dots, n$  and  $\Gamma = \text{diag}[\gamma_1, \dots, \gamma_p]$ . The sum in the second term of (39b) represents the scalar equality constraints given by the matrix equality in (37c). The polynomials  $s_1, s_2$  are the SOS-multipliers, resulting from the application of the Positivstellensatz, which certify the inequality constraints. The polynomials  $h_{lk}$  are indefinite multipliers certifying the equality constraints. The highest monomial degree in  $v(\bar{x}_0)$  is chosen to be equal to the highest monomial degree of  $v(\bar{x})$  in  $V(\bar{x})$ . As the constraint (39c) involves only the  $\bar{x}_0$  coordinates, the associated multiplier  $s_2$  contains polynomial terms only in  $\bar{x}_0$ .

The algorithm has bilinear terms in the SOS-multipliers and  $B_0$ , respectively  $Q_0$ . As is the case in the program in (35), we solve (39) iteratively.

**Remark 4** If  $\partial(V) = 2$  then the optimization can be performed to directly minimize  $\det(Q_0)^{1/n}$  without using the surrogate set. This removes the constraints (39c) and (39e) from the algorithm.

**Remark 5** In the case of  $\hat{\sigma} = 0$ , i.e. the covariance in the initial state is fixed to zero,  $\mathcal{R}_0$  can be obtained directly from the computed estimate  $\mathcal{R}$  by setting all terms containing variance modes to zero. In this case there is no need to solve (39).

**Remark 6** The complementary problem of maximizing the allowed covariance in the initial conditions for a fixed mean can be done by inserting the desired fixed matrix  $Q_0$  and moving  $\hat{\sigma}$  into the objective. The objective then consists of the convex expression  $\det(\hat{\sigma})^{1/n}$  and the resulting problem can be solved without the use of a surrogate set and its associated constraints.

## 5 Illustrative Examples

We demonstrate the proposed method for an uncertain Van-der-Pol (VDP) system and for the dynamics investigated in Iannelli *et al.* (2018). Both dynamics are affected by uniformly distributed uncertainty. While the first example has an uncertainty-independent equilibrium point, the second has an uncertainty-dependent equilibrium and thus converges to a set  $\mathcal{I}$ .

We denote in the following a uniform distribution between the boundary values  $u$  and  $v$  by  $\text{Unif}(u, v)$ . The choice of a uniform distribution is motivated here by the possibility to compare the results to previous studies. However, any other  $\mathcal{L}_2$ -distribution can be considered using the methods presented by simply changing the polynomial basis.

The numerical results were computed with Matlab 2018b, using the open-source toolbox YALMIP (Lofberg 2009) to formulate the SOS programs and the commercial solver Mosek to solve the SDPs. The scripts used to compute these examples can be found in <https://github.com/evaahbe/roa-analysis.git>

### 5.1 Uncertain Van-der-Pol dynamics

The first example consists of the VDP dynamics

$$\begin{aligned} \dot{x}_1 &= -x_2, \\ \dot{x}_2 &= -c(\xi)(1 - x_1^2)x_2 + x_1, \end{aligned} \quad (40)$$

where  $c(\xi) \sim \text{Unif}(0.7, 1.3)$  is a random variable depending on the stochastic germ  $\xi \sim \text{Unif}(-1, 1)$ . In order to obtain optimal convergence properties we use the Legendre polynomial basis for the PC expansion of the dynamics which is the basis associated with uniform probability distributions. The PC dynamics have an equilibrium point  $\bar{x}_{\text{EP}} = 0$  and thus the equilibrium set  $\mathcal{I}$  consists of the zero point which implies an uncertainty-independent equilibrium point of (40) at  $x_{\text{EP}} = 0$ . This equilibrium point is stable for  $c > 0$  and for any fixed  $c > 0$  the true

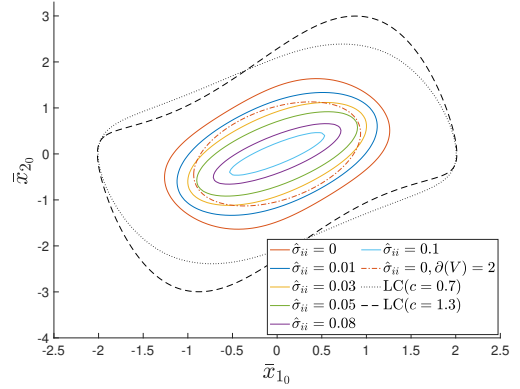


Fig. 1. Estimates of  $\mathcal{R}_0$  in terms of the initial mean states for various cases of fixed variance on the initial state. The results are obtained from a quartic  $V$  for several variance sizes and from a quadratic  $V$  for the case of zero minimum variance for comparison. An equal variance in both initial coordinate states is considered. The black dashed and dotted lines show the VDP limit cycle for the extreme realizations of  $c(\xi)$ .

ROA is given by a unstable limit cycle.

In order to choose the truncation order of the PC expansion which satisfies Assumption 1 we simulate the PC dynamics and truncate after the significant modes found at  $p = 3$ , resulting in a total of  $p + 1 = 4$  modes per dimension. This procedure is explained in more detail in Section 5.3. The sublevel set  $\mathcal{R}$  is obtained from the program (35) for  $\partial(V) = 4$ , and the results used to compute the ROA estimate  $\mathcal{R}_0$  as in (39) for different values of fixed variance on the initial condition. Thereby, a diagonal covariance matrix  $\hat{\sigma}$  with equal diagonal entries is chosen. The results are presented in Figure 1 (solid lines). As intuitively expected, the  $\mathcal{R}_0$  in terms of the initial state of the mean modes decreases with increasing size of variance in the initial state. Additionally, for comparison of different Lyapunov function degrees, the  $\mathcal{R}_0$  estimate with zero initial variance for a quadratic  $V$  is compute (red dash-dot line). Figure 1 shows how the higher degree  $V$  returns larger estimates of the ROA in this case. Further, Figure 1 reveals the true ROA of the stochastic system which here consists of the intersection of the two simulated limit cycles for the extreme realizations of  $c(\xi)$  (black lines).

### 5.2 Dynamics with uncertainty dependent equilibria

In the second example we consider the following uncertain dynamics studied in Iannelli *et al.* (2018),

$$\begin{aligned} \dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 + c(\xi), \\ \dot{x}_2 &= 3x_1 - x_2 - x_2^2, \end{aligned} \quad (41)$$

where  $c(\xi) \sim \text{Unif}(0.9, 1.1)$  with  $\xi \sim \text{Unif}(-1, 1)$ . This system has one stable and one unstable equilibrium point whose location is uncertainty-dependent. Also here we use the Legendre polynomial basis for the PC expansion of the system for which the simulation of a sample trajectory provides the exact location of the stable equilibrium point. From the validation procedure for Assumption 1 (see Section 5.3) we find that choosing  $p = 2$  captures the significant modes. With this the stable equilibrium



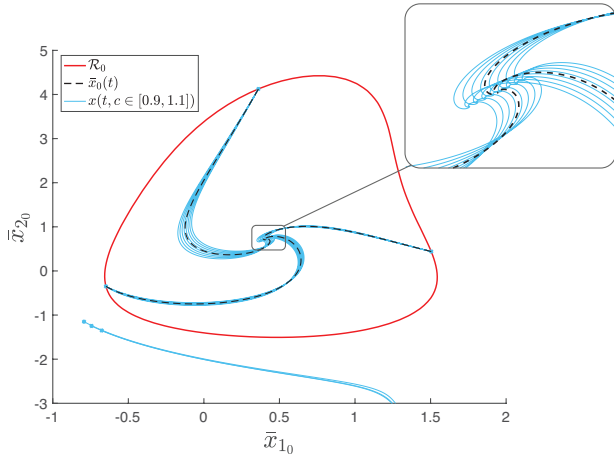


Fig. 2. The result for  $\mathcal{R}_0$  is shown in the red solid line. For various initial conditions on the boundary of  $\mathcal{R}_0$  a MC simulation of the system (41) for various realizations of the random variable is shown (blue lines). Note that as such the blue lines are each a trajectory of a deterministic system and since for deterministic variables the mean equals the nominal value,  $x_{\text{nom}} = m(x_{\text{nom}})$ , the trajectories are plotted here in the mean coordinates  $\bar{x}_0$  for consistency. The close-up shows how depending on the realization of uncertainty the trajectories converge to different equilibrium points, which together build the equilibrium set  $\mathcal{I}$  as given by Lemma 1. For three neighboring initial states with diverging trajectories the worst-case result of the MC simulation is plotted.

point is found to lie at  $\bar{x}_{\text{EP}} = [0.4086, 0.7145, 0.0369, 0.0456, -0.0005, -0.0012]$ .

As the PC expanded system has a non-zero equilibrium point, we obtain from Lemma 1 the equilibrium set  $\mathcal{I}$  of the stochastic system. The close-up in Figure 2 illustrates this set by showing how trajectories converge to a different equilibrium point for different realizations of the uncertainty.

The ROA estimate is computed from (35) for a quartic  $V$  and the results used to obtain the ROA  $\mathcal{R}_0$  in terms of the mean modes for zero variance on the initial state, as described in Remark 5. The results can be seen in Figure 2. The comparison with the ROA estimates in Iannelli *et al.* (2018) shows that the approach proposed here provides similar sizes of ROA. For illustration, three examples of a Monte-Carlo (MC) simulation starting at the boundary of the ROA and using each 8 realizations of the random variable over the distribution range are shown. The true ROA for this system is unknown. In order to obtain an upper bound on the conservatism of  $\mathcal{R}_0$  we search for diverging trajectories by performing MC simulations for a range of initial conditions located in the neighborhood outside of  $\mathcal{R}_0$ . The closest diverging trajectories found with the chosen sampling grid are shown in Figure 2.

### 5.3 Pre- and post validation of Assumption 1

The truncation order for a PC expansion needs to be decided such that Assumption 1 is satisfied. This can be achieved by simulating the dynamics of the PC expanded system for a high truncation order and a range of initial conditions in the region of interest. Since we consider second order random processes, the PC expansion converges in the  $\mathcal{L}_2$ -sense. Hence, there will be a truncation order for which the contribution from the higher

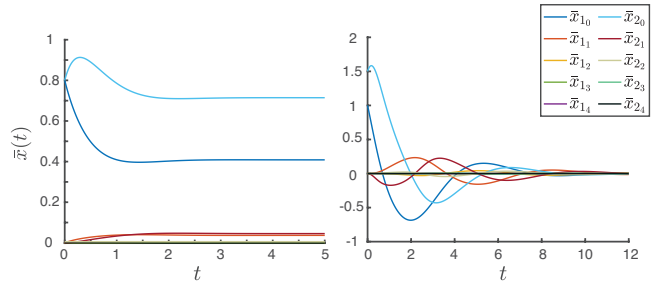


Fig. 3. *Right plot:* Evolution of the stochastic modes of (40) starting from the deterministic  $x_{\text{ini}} = [1, 1.5]$ . The modes for  $p \geq 4$  are found negligible. All modes eventually converge to zero, which is the equilibrium point. *Left plot:* Evolution of the stochastic modes of (41) starting from the deterministic  $x_{\text{ini}} = [0.8, 0.8]$ . Here, the modes for  $p \geq 3$  are found negligible. The coordinates of the attractive equilibrium point correspond to the limit values of each mode.

stochastic modes can in practice be considered negligible, and the truncation order is thus chosen such that only the significant modes are captured. For the example of the VDP, the significance of the first five stochastic modes has been investigated by simulating its PC dynamics for various initial conditions. Figure 3 shows the evolution of the modes starting from the deterministic initial condition  $x_{\text{ini}} = [1, 1.5]$ . It can be seen that the stochastic modes for  $p > 3$  are practically negligible. The same procedure was conducted for the second example, where Figure 3 reveals that choosing  $p = 2$  captures the significant modes.

Once the ROA estimates are calculated, a further validation of Assumption 1 is performed by verifying the computed ROA results for the true stochastic system. In the first example this is directly done by confirming in Figure 1 that the computed ROAs lie within the true ROA of the system given by the intersection of the black lines. For the second example, we ran an MC simulation of the stochastic system (41) for 1000 initial conditions on the boundary of the  $\mathcal{R}_0$ , considering for each of them 20 realizations of the uncertainty ranging over the distribution and confirmed their convergence to the equilibrium set.

### 5.4 Comments on the numerical implementation

The computational tractability of solving any SOS-program depends crucially on the size of the problem. The problem size scales exponentially in the number of states and polynomial degrees (polynomially, if scaled in either state or polynomial degree alone). While the PC expansion approach does not alter the polynomial degrees it does lead to a  $(p + 1)$ -fold increase of the number of states. Depending on the number of modes needed to represent the system with sufficient accuracy, the number of states can quickly become prohibitively large for low-dimensional stochastic systems. Research on more efficient SDP-solvers is ongoing and this limitation is likely to be alleviated in the future. One immediate remedy is offered by the DSOS/SDSOS framework introduced in Ahmadi and Majumdar (2019), which can solve SOS-programs tractably for up to 50 states. While potentially resulting in more conservative estimates these relaxations promise a significant speed up of the SOS program.

## 6 Conclusion

In this work we present a method to compute inner estimates of the region of attraction of stochastic nonlinear systems. The proposed method is applicable to a broad class of system consisting of second order random processes which are affected by uncertainties coming from any  $\mathcal{L}_2$ -distribution and which are further allowed to have uncertainty-dependent equilibria. The analysis is enabled by using Polynomial Chaos expansions through which a stochastic ODE is converted into a deterministic one. Using suitable stability notions in the form of moment boundedness and Lyapunov stability, it is shown how the ROA analysis of the PC expanded system offers direct information on the attractive behavior of the stochastic system for which a notion of a ROA is derived. A numerical implementation for obtaining inner estimates of the ROA when the PC expanded system has a polynomial expression are provided via SOS optimization. The application to two examples taken from the literature shows that the proposed approach provides estimates of the ROA which are comparable to literature results obtained with less general methods. The analysis method proposed here can be used and extended for various purposes among which are the stability analysis of systems with more complex equilibrium behavior, and the use of stochastic ROA analysis in controller design.

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