

An equilibrium-independent region of attraction formulation for systems with uncertainty-dependent equilibria

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Abstract—This paper aims to compute the region of attraction (ROA) of equilibrium points whose location is modified by the uncertainties. The local stability region is formulated as an equilibrium-independent level set by restricting the attention to contractive functions which do not explicitly depend on the equilibrium. Another favourable feature of the approach is that it can be applied to systems having one or more branches of steady-state solutions (e.g. multistable systems). Inner estimates of the ROA are numerically computed by means of Sum of Square techniques, which allow to specify the allowed uncertainty range and the analyzed branch as set containment conditions, resulting in a compact and flexible formulation. A numerical example shows the application of the method and highlights its peculiar features.

I. INTRODUCTION

The Region of Attraction (ROA) of an equilibrium point (or fixed point) x^* is the set of all the initial conditions from which the trajectories of the system converge to x^* as time goes to infinity [1]. The characterization of this region of the state space is of practical interest to guarantee the safe operation of nonlinear systems. For this reason, a number of studies have been devoted to it [2], for the most part building on the fact that Lyapunov functions level sets are contractive and invariant [1]. Improvements aimed at reducing the conservatism of these inner estimates [3] and providing with more efficient computational strategies [4] have also been proposed.

The case when the system is subject to uncertainties has received less attention, even though various authors have proposed strategies to tackle this problem. For example, in [5] parameter-dependent rational Lyapunov functions are proposed, whereas in [6] inner estimates are provided with a branch-and-bound algorithm aimed at mitigating the conservatism associated with parameter-independent Lyapunov functions.

Approaches to estimate robust ROAs typically assume the equilibrium point is independent of the uncertainty. This assumption is primarily for the sake of simplicity and in most cases the uncertainty does alter the equilibrium point. Indeed allowing for a variation in the equilibria would imply using parameter-dependent Lyapunov functions at the expense of

increased computational complexity. Adding to this, it may not be possible to obtain a closed form expression of the equilibria as a function of the uncertainties, therefore the parameterization of suitable Lyapunov functions would not be practical.

This paper, which focuses on real constant uncertainties, addresses this issue by proposing a framework for region of attraction analysis of uncertainty-dependent equilibrium points. Inner estimates of the ROA are formulated as invariant level sets of a contractive function V , which does not explicitly depend on the equilibrium point or on the uncertainties. The result thus consists in an Equilibrium-Independent Region of attraction (EIR). This is a central feature of the formulation as described in Section III and demonstrated via a numerical example in Section IV.

The idea of guaranteeing a property of the system without knowledge of the equilibrium is inspired by the work in [7], where the notion of equilibrium-independent passivity was first introduced. This was then generalized in [8], where the concept of equilibrium-independent dissipativity for systems with unknown equilibria was formulated. However, the work here considers an equilibrium-independent storage function, and, equally important, allows to study systems with multiple equilibria, relaxing the assumption in [7] that there exists a unique equilibrium for a given perturbation. This key feature is achieved by specifying the branch of equilibria considered in the analyses as the level set of a suitable function. If no a priori knowledge is available, a candidate function can be identified via Sum of Squares (SOS) techniques. This optimization tool will also be used to provide computational recipes for estimating subsets of the EIR.

A similar problem was studied in [9], where lower and upper estimates of the ROA were formulated based on viability theory. Specifically, that work considered convergence of the trajectories to a given set, whereas here the problem of asymptotic converge to an (unknown) equilibrium point is investigated. The paper is also connected to the work in [10], where an algorithm based on contraction metrics was proposed to relax the uncertainty-independent equilibrium hypothesis. While that study focused on global stability certificates, this paper is concerned primarily with region of attraction estimation and tailors the approach to this application (e.g. enhancing the importance of having an equilibrium-independent formulation), even though it could be employed for certifying global stability of branches of fixed points as well.

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II. BACKGROUND

A. Problem statement

Consider an autonomous nonlinear system of the form

$$\dot{x} = f(x), \quad x(0) = x_0 \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the vector field. The vector $x^* \in \mathbb{R}^n$ is called a *fixed* or *equilibrium* point of (1) if $f(x^*) = 0$. Let $\phi(t, x_0)$ denote the solution of (1) at time t with initial condition x_0 . The ROA associated with an attractive equilibrium x^* is

$$\mathcal{R} := \{x_0 \in \mathbb{R}^n : \lim_{t \rightarrow \infty} \phi(t, x_0) = x^*\} \quad (2)$$

Thus \mathcal{R} is the set of all initial states that converge to x^* .

Consider now the system governed by

$$\dot{x} = f(x, \delta) \quad (3)$$

where $\delta \in \Delta \subset \mathbb{R}^{n_\delta}$ is the vector of constant unknown parameters, Δ is a known bounded set, and $f : \mathbb{R}^n \times \Delta \rightarrow \mathbb{R}^n$. The robust Region of Attraction \mathcal{R}_δ is defined as the intersection of the ROAs for all systems governed by (3). Note that the equilibrium point x^* can, in general, depend on the uncertainties as explicitly reported in the definition of \mathcal{R}_δ given next

$$\mathcal{R}_\delta := \bigcap_{\delta \in \Delta} \{x_0 \in \mathbb{R}^n : \lim_{t \rightarrow \infty} \phi(t, x_0, \delta) = x^*(\delta)\} \quad (4)$$

where $\phi(t, x_0, \delta)$ denotes the solution of (3) at time t with initial condition x_0 and subject to δ . This work will focus on inner estimates of \mathcal{R}_δ as defined in (4).

B. Sum of Squares

Let us denote, for $x \in \mathbb{R}^n$, the set of all polynomials in n variables with $\mathbb{R}[x]$. For $q \in \mathbb{R}[x]$, $\partial(q)$ denotes the degree of q . A polynomial q is said to be a Sum of Squares (SOS) if there exists a finite set of polynomials q_1, \dots, q_k such that $q = \sum_{i=1}^k q_i^2$. The set of SOS polynomials in x is denoted by $\Sigma[x_1, \dots, x_n]$, abbreviated here with $\Sigma[x]$. The importance of SOS polynomials is due to their connection with convex optimization [11]. Namely, $q \in \Sigma[x]$ if and only if there exists $Q = Q^T \succeq 0$ such that $q = z^T Q z$, where z is a vector gathering the monomials of q of degree less than or equal to $\partial(q)/2$. This problem can be recast as a semidefinite program and there are freely available software toolboxes to solve this in an efficient manner. In this work, the software SOSOPT from the suite of libraries [12] will be used.

One of the applications of SOS optimization is finding polynomial functions that satisfy set containment conditions. Specifically, the following property, which can be obtained as application of the Positivstellensatz (P-satz) Theorem, is central to the purposes of the article

Lemma 1: [11] Given $h, f_0, \dots, f_r \in \mathbb{R}[x]$, the following set containment holds

$$\{x : h(x) = 0, f_1(x) \geq 0, \dots, f_r(x) \geq 0\} \subseteq \{x : f_0(x) \geq 0\} \quad (5)$$

if there exist *multipliers* $r \in \mathbb{R}[x]$, $s_1, \dots, s_r \in \Sigma[x]$ such that

$$r(x)h(x) - \sum_{i=1}^r s_i(x)f_i(x) + f_0(x) \in \Sigma[x] \quad (6)$$

III. EQUILIBRIUM INDEPENDENT ROA

A. Problem formulation

The framework proposed in this paper to determine inner estimates of the robust ROA takes the cue from the concept of equilibrium-independent passivity, introduced in [7]. The objective here is to find a storage function whose level set is attractive and invariant for any potential equilibrium point of the uncertain system. There are two main distinctions relative to [7]. First, the storage function does not explicitly reference the equilibrium point. This yields an equilibrium-independent ROA. Second, the proposed approach does not assume the existence of a unique equilibrium.

First, let us define $\mathcal{E} := \{x^* : f(x^*, 0) = 0\}$, i.e. \mathcal{E} is the set of equilibria of the *nominal* vector field (for the sake of notation, we assume without loss of generality that $\delta = 0 \in \Delta$ corresponds to nominal conditions). We will assume that \mathcal{E} is nonempty, with $|\mathcal{E}|$ indicating the cardinality of the set. In general, $|\mathcal{E}| \geq 1$, i.e. the nominal vector field has one or more fixed points x^* . Specifically, we will be concerned with the subset $\mathcal{E}^a \subseteq \mathcal{E}$ gathering attractive equilibria.

Define the induced set $\mathcal{D}_{x^*}^{\mathcal{E}} := \{\bar{x} : 0 = f(\bar{x}, \delta), \forall \delta \in \Delta\}$ which associates each element x^* of \mathcal{E}^a with the branch of equilibria resulting from varying δ inside Δ . Note that the definition of $\mathcal{D}_{x^*}^{\mathcal{E}}$ only assumes that, as δ takes values in the uncertainty set, the branch consists of attracting equilibria. Further discussions on this aspect are referred to Remark 2 below. The objective is to compute an equilibrium independent region of attraction for the branch of fixed points \bar{x} around the selected nominal equilibrium point x^* .

We could attempt to find a parameter-dependent Lyapunov function $V(x, \delta)$ to estimate the ROA for $\bar{x}(\delta)$. This is the existing approach, but it is computationally demanding since the variables δ need to be included in the SOS program. Moreover, it entails having a closed form expression for the dependence of the equilibria on the uncertainties in order to ensure that $V(\bar{x}(\delta), \delta) = 0$.

To avoid this, we consider the coordinate transformation $y(x, \delta) = x - \bar{x}(\delta)$. The new coordinate y can be interpreted as the deviation of the state x relative to the equilibrium point $\bar{x}(\delta)$ for uncertainty $\delta \in \Delta$. It is stressed that, since δ is constant, it holds $\dot{y} = \dot{x}$. Hence the vector field $f(y + \bar{x}(\delta), \delta)$ governs the evolution of the state x relative to the equilibrium point $\bar{x}(\delta)$ that occurs at δ .

We are now ready to state the following result, which gives conditions to determine invariant and attractive regions associated with the equilibria of (3) via equilibrium-independent Lyapunov functions.

Theorem 1: Let x^* be an attractive equilibrium point of the nominal system, i.e. $x^* \in \mathcal{E}^a$. We denote with $\mathcal{D}_{x^*}^{\mathcal{E}}$ the associated set and we apply to f the coordinate transformation $x = y + \bar{x}$, with $\bar{x} \in \mathcal{D}_{x^*}^{\mathcal{E}}$.

If there exists a smooth, continuously differentiable function $\tilde{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ and an associated level set $\Omega_{\tilde{V}, \gamma} = \{y \in$

$\mathbb{R}^n : \tilde{V}(y) \leq \gamma\}$ such that

$$\tilde{V}(0) = 0 \quad \text{and} \quad \tilde{V}(y) > 0 \quad \forall y \setminus \{0\} \quad (7a)$$

$$\nabla \tilde{V}(y) \cdot f(y + \bar{x}(\delta), \delta) < 0 \quad \forall y \in \Omega_{\tilde{V}, \gamma} \setminus \{0\}, \quad (7b)$$

$$\forall \delta \in \Delta, \forall \bar{x} \in \mathcal{D}_{x^*}^{\mathcal{E}}$$

Then $\Omega_{\tilde{V}, \gamma}$ is an inner estimate of the EIR of the fixed points $\bar{x} \in \mathcal{D}_{x^*}^{\mathcal{E}}$ when (3) is subject to perturbations in Δ .

Proof: The theorem assumes there exists a function \tilde{V} which is positive and decreasing inside a level set $\Omega_{\tilde{V}, \gamma}$, and its only zero is the origin. This implies, by application of the Lyapunov direct method [1], that this region is attractive and invariant, therefore it holds $\Omega_{\tilde{V}, \gamma} \subseteq \mathcal{R}_\delta$. Note the subtlety, compared to standard Lyapunov function level sets, that the function \tilde{V} decreases along trajectories y representing the deviation of the original state x from the equilibrium point \bar{x} as δ take values in the uncertainty set. These trajectories will ultimately converge to $y = 0$, that is the state x will settle down to the equilibrium. ■

Remark 1: The robust Region of Attraction provided by Theorem 1 is equilibrium independent, due to the fact that the level set is a function of y only. This choice privileges the interpretation of the results by making the domain of attraction only implicitly related to \bar{x} , which is in principle unknown (due to its dependence on the uncertainty). In view of this, an apparent practical benefit of the EIR formulation is that it is not required to fix a value of \bar{x} to represent the predicted region of local stability. In fact, this approach underlies the concept of *family* of ROA, consisting in a single region $\Omega_{\tilde{V}, \gamma}$ which has embedded the information relative to multiple domains of attraction. Indeed if \tilde{V} satisfies the conditions of Theorem 1, then $V(x, \delta) = \tilde{V}(x - \bar{x}(\delta))$ provides a parameter-dependent Lyapunov function and $\Omega_{V, \gamma} = \{x \in \mathbb{R}^n : V(x, \delta) \leq \gamma\}$ is a ROA relative to the equilibrium point $\bar{x}(\delta)$. For similar reasons, a parameter-dependent ROA of the type $V(y, \delta)$ has not been pursued. This choice, despite being in principle a source of conservatism, aims to reduce the computational complexity of the ensuing numerical problem and favour a better interpretation of the results as commented above.

Remark 2: The proposed framework can be applied to systems having multiple equilibria. This is achieved by introducing the set \mathcal{E}^a (where $|\mathcal{E}^a| \geq 1$) and associating an EIR to *each* $x^* \in \mathcal{E}^a$. The induced set $\mathcal{D}_{x^*}^{\mathcal{E}}$ is then instrumental in Theorem 1 to ensure that the analyses are relative to a specific branch of solutions only. In practice, it is made the additional hypothesis that $f(\bar{x}, \delta) = 0$ has a unique solution $\bar{x}(\delta)$, associated with a given x^* , for $\delta \in \Delta$. This assumption is not restrictive and, to show this, we consider the Implicit Function Theorem (IFT) [13]. Under the condition that the linearized Jacobian of f around the pair $(x^*, 0)$ is not singular, it is guaranteed existence and uniqueness of a mapping $F : \bar{x} = F(\delta)$ in neighbourhoods D and X of $\delta = 0$ and x^* respectively. This condition is always satisfied if the steady-state solutions of f do not undergo qualitative changes (e.g. bifurcations) when the uncertainty vary inside the set Δ [14]. Having in mind that

this framework is employed to study ROA of attractive fixed points, it can be assumed that $D = \Delta$ holds, i.e., the additional assumption is natural in this context. Note that Theorem 1 still holds when the IFT is not verified (e.g., pitchfork bifurcation, where it suffices to consider one of the stable emanating branches), but the computational recipes proposed in the next sections do not apply.

B. Equilibria's region identification

The computation of the EIR with Theorem 1 involves finding a function that satisfies set containment conditions. Since interest is restricted to polynomial vector fields, the problem can be recast as an SOS program building on the result of Lemma 1.

In this discussion \bar{x} and δ are algebraic indeterminates satisfying particular conditions, e.g. $\delta \in \Delta$. The bounded set Δ is described in this work as a semialgebraic set [15]

$$\Delta = \{\delta \in \mathbb{R}^{n_\delta} : m_i(\delta) \geq 0, i = 1, \dots, j\} \quad (8)$$

where the functions m_i are polynomials in δ , whose definition depend on the type of uncertainties featuring the system. This work will consider parametric uncertainties, and we will denote $\underline{\delta}_i$ and $\bar{\delta}_i$ respectively the minimum and maximum values for each component δ_i of δ . Then, a polynomial m_i can be associated with each parameter

$$m_i(\delta_i) = -(\delta_i - \underline{\delta}_i)(\delta_i - \bar{\delta}_i) \quad (9)$$

$$\delta \in \Delta \iff m_i(\delta_i) \geq 0$$

Another set containment prescribed by (7b) is that the inequality must hold $\forall \bar{x} \in \mathcal{D}_{x^*}^{\mathcal{E}}$, that is it has to be fulfilled by all the equilibria \bar{x} on the analyzed branch. In order to incorporate this condition into an SOS program, the idea is to find a function g such that $\bar{x} \in \mathcal{D}_{x^*}^{\mathcal{E}} \implies g(\bar{x}) \geq 0$ and, similarly to what is done with the uncertainties, use this function to constrain the variables \bar{x} . If no a priori knowledge on the location of the branch is available to define g , the following strategy can be followed to compute it.

A function g_d with a prescribed shape is considered, and the smallest positive scalar $c_1 > 0$ such that $\bar{x} \in \mathcal{D}_{x^*}^{\mathcal{E}} \implies \bar{x} \in \Omega_{g_d, c_1}$ is computed. Possible choices for g_d are the spheroid or ellipsoid centered in x^* . Then, $g = c_1 - g_d$ is a candidate function to enforce the constraint. This problem can be recast as an SOS optimization. In order to specialize the search of the function g to the branch corresponding to \bar{x}^0 , the auxiliary level set Ω_{p_b, R_b} is introduced. The rationale behind the selection of p_b and R_b is that Ω_{p_b, R_b} should include the branch of equilibria associated to \bar{x} , i.e. $\mathcal{D}_{x^*}^{\mathcal{E}} \subseteq \Omega_{p_b, R_b}$.

Program 1:

$$\min_{s_b, s_{d_i} \in \Sigma[\bar{x}; \delta]; r_f \in \mathbb{R}[\bar{x}; \delta]} c_1$$

$$r_f f - s_b(R_b - p_b) - \Gamma(\delta) + (c_1 - g_d) \in \Sigma[\bar{x}, \delta] \quad (10a)$$

$$\Gamma(\delta) = s_{d_1} m_1(\delta_1) + \dots + s_{d_i} m_i(\delta_i) + \dots + s_{d_j} m_j(\delta_j) \quad (10b)$$

This SOS program is a direct application of Lemma 1. The last term in (10a) is equivalent to f_0 in (6) (i.e. the *including* set function), whereas the others are the *included* sets (h, f_i) and relative multipliers. The first term provides

the constraint that \bar{x} is a fixed point of the original vector field f as δ varies within its range. Note however that, if $|\mathcal{E}| > 1$ this information is not sufficient because there are multiple branches of solutions (\bar{x}, δ) for which $f(\bar{x}, \delta) = 0$. For this reason, the aforementioned set Ω_{p_b, R_b} is employed. Despite being seemingly similar, Ω_{p_b, R_b} and Ω_{g_d, c_1} have a substantially different meaning. The latter is indeed the smallest level set (for a given shape) to include the equilibria of the system, whereas Ω_{p_b, R_b} has the purpose of selecting the branch of interest. Note that R_b has to be chosen such that Ω_{p_b, R_b} does not include other branches of solutions. The optimization over c_1 guarantees then, even without a good initial guess for R_b , the sought estimation for Ω_{g_d, c_1} is eventually obtained (e.g. iteratively).

Finally, Γ guarantees that the SOS constraint hold in the uncertainty set by applying the rationale discussed in (9).

Remark 3: The aforementioned approach is proposed here to characterize the set $\mathcal{D}_{x^*}^{\mathcal{E}}$. In other words, it allows to identify regions of the state space where there is a branch of fixed points. It is common in the study of nonlinear dynamics to relate properties of the system (e.g. bifurcations in the steady-state solutions) to algebraic conditions fulfilled by the Jacobian of the vector field and its higher derivatives [16]. These can be easily appended to Program 1, and thus the approach proposed in this work could be applied to investigate additional properties of uncertain systems described by the vector field in (3). With this regard, note that an alternative strategy where also the shape of the level set function g_d is optimized (other than the size) can be easily devised with a straightforward extension of Program 1.

C. An algorithm for inner estimates of the EIR

Based on the foregoing discussion, we now formulate an SOS program to determine equilibrium independent regions of attraction based on Theorem 1. The independent variables of the optimization are denoted by $\tilde{y} = [y; \bar{x}; \delta]$.

Program 2:

$$\begin{aligned} & \max_{s_D, s_{d_i}, s_1 \in \Sigma[\tilde{y}]; r_f \in \mathbb{R}[\tilde{y}]; \tilde{V} \in \mathbb{R}[\tilde{y}]} \gamma \\ & \tilde{V} - L_1 \in \Sigma[\tilde{y}] \\ & r_f f - s_1(\gamma - \tilde{V}) - s_D g - \Gamma - (\nabla \tilde{V} f + L_2) \in \Sigma[\tilde{y}] \end{aligned} \quad (11)$$

where Γ is defined as in (10b) and L_1 and L_2 are positive definite polynomials, specifically $L_1 = \epsilon_1 y^T y$ and $L_2 = \epsilon_2 \tilde{y}^T \tilde{y}$ with ϵ_1 and ϵ_2 small real numbers on the order of 10^{-6} . These guarantee that the SOS constraints in (11) are sufficient conditions for (7).

This problem features the bilinear terms $s_1 \gamma$ and $s_1 \tilde{V}$. While the first involves the objective function γ and thus can be overcome via cost bisection [17], the term $s_1 \tilde{V}$ makes the above program non-convex. For this reason, an iterative scheme, inspired by the V - s iteration from [18], which allows to solve Program 2 via a series of convex problems, is proposed.

Algorithm 1:

Output: the level set $\Omega_{\tilde{V}, \gamma}$ (inner estimate of the EIR).

Input: a polynomial \tilde{V}^0 satisfying (7) for some γ .

$$\begin{aligned} & 1) \gamma - \text{Step} \\ & \gamma^* = \max_{s_1, s_D, s_{d_i} \in \Sigma[\tilde{y}]; r_f \in \mathbb{R}[\tilde{y}]} \gamma \\ & r_f f - s_1(\gamma - \tilde{V}^0) - s_D g - \Gamma - (\nabla \tilde{V}^0 f + L_2) \in \Sigma[\tilde{y}] \\ & 2) \beta - \text{Step} \\ & \beta^* = \max_{s_2 \in \Sigma[\tilde{y}]} \beta \\ & (\gamma^* - \tilde{V}^0) - s_2(\beta - p) \in \Sigma[\tilde{y}] \\ & 3) V - \text{Step} \\ & \tilde{V} - L_1 \in \Sigma[\tilde{y}] \\ & (\gamma^* - \tilde{V}) - \bar{s}_2(\beta^* - p) \in \Sigma[\tilde{y}] \\ & r_f f - \bar{s}_1(\gamma - V) - s_D g - \Gamma - (\nabla \tilde{V} f + L_2) \in \Sigma[\tilde{y}] \end{aligned}$$

In essence, in the γ -step the function \tilde{V} is held fixed to the value \tilde{V}^0 calculated at the end of the previous iteration and the multiplier s_1 is optimized. The resulting polynomial \bar{s}_1 is then employed in the V -step, where the shape of the level set function is optimized. The β step has the goal to determine the largest value of the scalar β such that $\Omega_{p, \beta} \subseteq \Omega_{\tilde{V}^0, \gamma^*}$, with the goal of enlarging the set $\Omega_{\tilde{V}, \gamma}$ computed in the V -step. p can be chosen as an ellipsoid based on the importance of specific directions in the state space. Note finally that a candidate function V is required to initialise the algorithm. A possible choice is any quadratic Lyapunov function proving asymptotic stability of the linearised nominal system (i.e., with respect to x^*). Due to the non-convexity of Program 2, the choice of \tilde{V}^0 has influence on the results and thus reinitialization of Algorithm 1 from previous solutions is a viable strategy to improve the predictions.

IV. NUMERICAL EXAMPLE

A. System description

The following nonlinear dynamics with an uncertain scalar parameter δ is considered

$$\begin{aligned} \dot{\phi} &= -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 + \delta \\ \dot{\psi} &= 3\phi - \psi - \psi^2 \\ \delta &\in [0.9; 1.1] \end{aligned} \quad (12)$$

The steady-state solutions of the system consist of two distinct branches of equilibria. By linearizing the Jacobian of (12) around the equilibria and evaluating the corresponding eigenvalues, it can be seen that one branch corresponds to sinks (i.e. stable fixed points), while the other corresponds to sources (i.e. unstable fixed points). Recalling the nomenclature introduced in Section III-A, this means that the cardinality of \mathcal{E} is 2 and thus the set of interest $\mathcal{D}_{x^*}^{\mathcal{E}}$ (i.e. the one associated to sinks) must be specified. This characterization can be performed by means of Program 1, where the fixed shape function g_d is taken here as the circle centred in the nominal fixed point (i.e. $\delta = 1$). Fig. 1 displays the set $\mathcal{D}_{x^*}^{\mathcal{E}}$ (dotted curve *Equilibria*) and the level set Ω_{g_d, c_1} (dashed curve g_d) for the branch of sinks

considered later in the EIR analyses. It can be noted that the level set circumscribes the equilibria, and thus is a good candidate to characterize $D_{x^*}^E$.

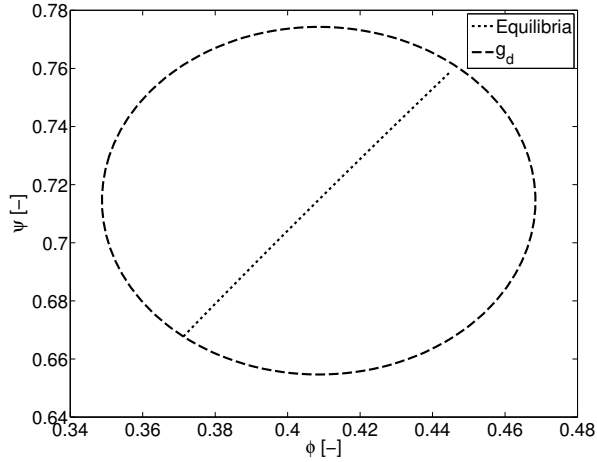


Fig. 1. Characterization of the set of sinks via level sets.

Another important feature of the dynamics in (12) is that the sinks are only locally asymptotically stable. Therefore, the approach presented in Section III can be used to determine the associated region of attraction. Note that, since the other branch of solutions is unstable, it is fundamental to specify that the analyses consider only the set of sinks, because otherwise no region of attraction could be estimated.

B. EIR of the sinks

Equilibrium-independent estimations of the ROA of system (12) are computed in this section by means of Algorithm 1. Fig. 2 shows the results obtained with functions \tilde{V} of different degree, specifically $\partial(\tilde{V}) = 2$ and $\partial(\tilde{V}) = 4$.

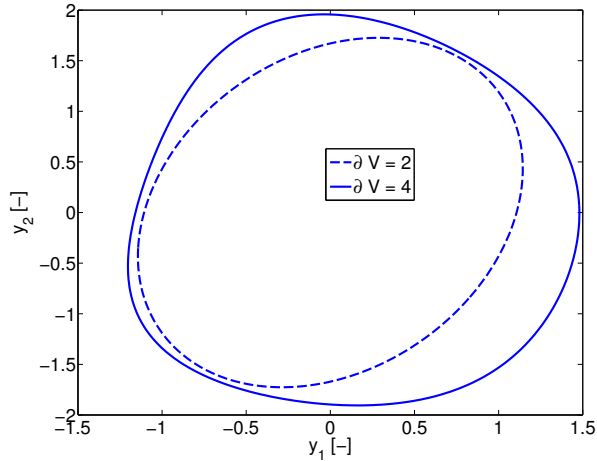


Fig. 2. EIR for $\partial(\tilde{V}) = 2$ and $\partial(\tilde{V}) = 4$.

Note first that the shifted coordinates $y_1 = \phi - \bar{\phi}$ and $y_2 = \psi - \bar{\psi}$ are reported on the axes of the plots. This is consistent with the central idea of the approach of representing the domain of attraction as an equilibrium-independent level set. It can also be ascertained that by increasing the degree of

\tilde{V} , the estimation of the ROA is improved, at the cost of a greater run time for Algorithm 1, which is the typical trade-off accuracy/complexity arising in SOS applications. It is interesting to register an enlargement of the stability region in the lower right part of the plot. Further aspects associated with the concept of EIR are investigated next.

In Remark 1 it was introduced the equilibrium-dependent level set $\Omega_{V,\gamma}$ which is associated with $\Omega_{\tilde{V},\gamma}$ via coordinates transformation. Once a value for the uncertainty is specified, $\Omega_{V,\gamma}$ provides an estimate of the ROA for the associated equilibrium point $\bar{x}(\delta)$. Taking the cue from this, a Monte Carlo-based search to quantify the conservatism associated with the predictions reported in Fig. 2 is employed. The goal is to compute the smallest $\gamma_f > \gamma$ (and associated δ) such that there is an initial condition x_0 on the boundary of Ω_{V,γ_f} for which the system does not eventually converge to the equilibrium point.

Results are commented next with regard to the estimations shown in Fig. 2 for the case of quartic V (serving as lower bound). The Monte Carlo search reveals that $\delta = 0.9$ is the critical (i.e. leading to the smallest γ_f) value for the uncertain parameter. Fig. 3 shows lower (LB) and upper (UB) bounds of the ROA, as well as cross markers corresponding to initial conditions on Ω_{V,γ_f} whose trajectories do not converge to the equilibrium (square marker).

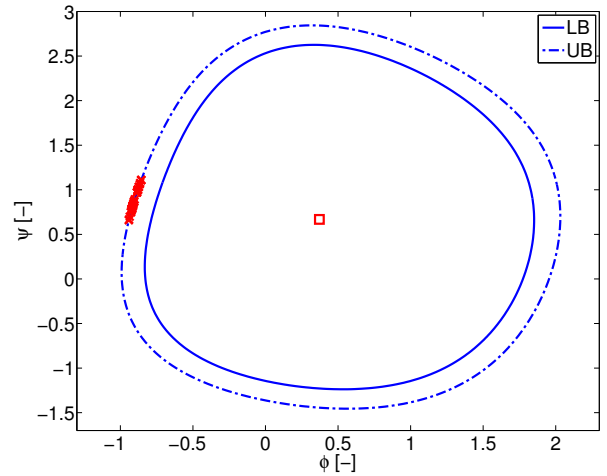


Fig. 3. Level sets $\Omega_{V,\gamma}$ and Ω_{V,γ_f} associated to $\delta = 0.9$.

It is interesting to notice that the cross markers in Fig. 3 are all distributed in the region where Fig. 2 featured the smaller gap between the curves $\partial(V) = 2$ and $\partial(V) = 4$ (note that the transformation $y = x - \bar{x}$ is implied by the comparison). This is an important observation as it shows that Algorithm 1 effectively exploits the higher degree of the level set function to enlarge the ROA in regions of the state space associated with converging trajectories.

The outcome in Fig. 3 can be interpreted as a *worst-case* analysis, in that it detects the closest points to the estimated set which do not belong to the ROA of the system, and the associated combination of uncertain parameters (in this case a scalar). One of the advantages of the equilibrium-

independent framework proposed in this work is that also the *degradation* of local stability in the face of uncertainties can be efficiently visualized. This can be done by taking advantage of the concept of family of ROA underlying the formulation, which allows to represent the results compactly in the y space (or y_1 - y_2 plane, in the particular case $n = 2$ considered here). The idea is to apply the Monte Carlo search discussed before repeatedly and so obtain γ_f as a function of the uncertainties. In the scalar case ($n_\delta = 1$), this is straightforward, but the strategy can in principle be applied to the case $n_\delta > 1$ by partitioning the uncertainty set Δ and associating with each cell a value of γ_f .

Fig. 4 shows an application of this approach, where the quartic level set $\Omega_{V,\gamma}$ from Fig. 2 is again used as basis for the analyses (LB). Based on the Monte Carlo campaign discussed above, a range of δ is associated with a level set if, within that uncertainty interval, there are diverging trajectories with initial conditions on its boundary.

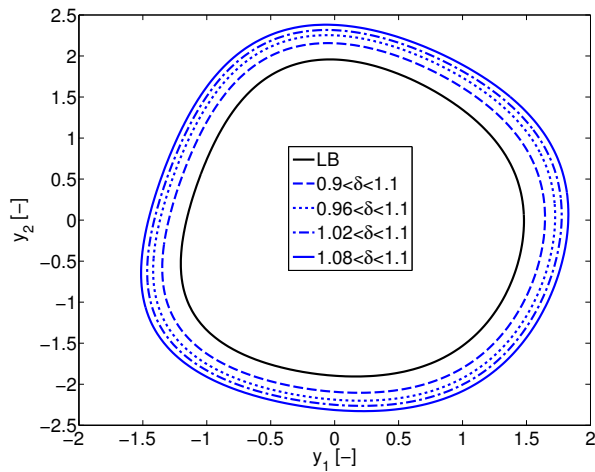


Fig. 4. Degradation of the local stability in the face of uncertainties.

These analyses can be interpreted as an equilibrium-independent upper bound estimation of the ROA for different uncertainty ranges. In other words, the results give a measure of the robustness of the local stability. The key feature is that this information is unrelated to the specific equilibrium point, i.e. it allows to isolate the effect of δ on the degradation of local stability from that on the equilibrium point. This is deemed an important feature, because it recovers the type of investigation commonly performed when the uncertainty-independent hypothesis is assumed for the equilibrium point. It is worth observing that, when the parameter-dependent level set $\Omega_{V,\gamma}$ is considered, these types of analysis would not be possible because the representation of the ROA is inherently connected to a specific equilibrium \bar{x} (and thus to a specific δ).

Note finally that, to complement the stability degradation study displayed in Fig. 4, an equilibrium-independent lower bound study can be performed by running Algorithm 1 for different uncertainty ranges.

V. CONCLUSION

This paper studied a framework for region of attraction analysis of uncertain systems whose equilibrium points are a function of the uncertainties. The key feature of the approach is that the domain of attraction is formulated as an equilibrium-independent level set, recovering in so doing the standard representation available for uncertain systems with uncertainty-independent equilibria. Another advantage is that the approach can be applied to systems with multiple equilibria. This is done by restricting the analyses to one branch of solutions at a time via set containments. An algorithm for the numerical calculation of estimates of the ROA is proposed, which recasts the problem as an SOS optimization.

The prowess of the proposed framework is demonstrated on a numerical example, consisting of a system with a branch of stable fixed points and a branch of unstable ones. Conservatism associated with the predictions is investigated, and a strategy which exploits the equilibrium-independent formulation is proposed to quantify the degradation of local stability in the face of uncertainties.

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