Feedback Control Design Maximizing the Region of Attraction of Stochastic Systems Using Polynomial Chaos Expansion ⋆

Eva Ahbe * **Petr Listov** ** **Andrea Iannelli** * **Roy S. Smith** *

* *Automatic Control Laboratory, Swiss Federal Institute of Technology (ETH Zurich), Switzerland,* {ahbee,iannelli,rsmith}@control.ee.ethz.ch.
** *Automatic Control Laboratory, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland,* {petr.listov}@epfl.ch.

Abstract: A feedback control design is proposed for stochastic systems with finite second moment which aims at maximising the region of attraction of the equilibrium point. Polynomial Chaos (PC) expansions are employed to represent the stochastic closed loop system by a higher dimensional set of deterministic equations. By using the PC expanded system representation, the available information on the uncertainty affecting the system explicitly enters the control design problem. Further, this allows Lyapunov methods for deterministic systems to be used to formulate the stability criteria certifying the region of attraction. These criteria are parametrized by the feedback gain and formulated in a polynomial optimization program which is solved using sum-of-squares methods. This approach offers flexibility in the choice of the stochastic feedback law and accounts for input constraints. The application is demonstrated by two numerical examples.

Keywords: Stochastic Nonlinear Systems, Feedback Control Design, Region of Attraction, Sum-of-Squares

1. INTRODUCTION

The region of attraction (ROA) of a nonlinear system is defined as the set of initial conditions from which system trajectories converge to an attractive equilibrium point. The size of the ROA is of interest in many application as it measures the robustness of the system with respect to perturbations in the initial conditions and defines the part of the state space in which a system can be safely operated. Therefore, it is of interest to investigate the design of feedback controllers which are able to enlarge the ROA of the operating point. This poses two major challenges for the control design. Firstly, conditions on the controller need to be formulated in a way such that an increase of the attractive properties of the system is obtained. And secondly, means to measure the size of the ROA are required in order to quantify the attractive region. For deterministic polynomial systems with affinely appearing control inputs, such feedback controllers have been previously proposed, e.g., in Jarvis-Wloszek et al. (2005), Chesi (2004), Majumdar et al. (2013). Both the feedback gains as well as the size of an inner estimate of the ROA are thereby obtained by formulating Lyapunov stability arguments in an optimization problem which aims to maximize the size of the ROA estimate (or a surrogate of it). Following an approach proposed in Parrilo (2000), these optimization problems can be relaxed to sum-of-squares (SOS) programs and solved with semidefinite programming techniques.

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Employing feedback control to increase the size of the ROA can be particularly desirable when uncertainties affect the system and exert a detrimental effect on the ROA. The task, in particular measuring the ROA, becomes significantly more complex in the case of uncertain systems. While the computation of inner estimates of the ROA was proposed for systems with uniformly distributed uncertainty (Topcu and Packard (2009); Iannelli et al. (2019); Valmorbida and Anderson (2017)), none of these methods include a control design aiming at enlarging the ROA.

In this work we consider stochastic systems in form of second order processes which can be affected by uncertainties coming from any probability distribution with bounded second moment. This class of systems represents most processes of the real world (Xiu and Karniadakis (2003)). For these systems we propose a control design which explicitly takes into account the statistical information available on the uncertainty and efficiently maximizes the ROA while also enforcing input constraints.

The proposed approach leverages the framework of Polynomial Chaos (PC) which is made possible by limiting the scope to second order processes. The framework enables the representation of a stochastic system affected by uncertain parameters of known second order distributions by a higher dimensional set of deterministic equations (see, e.g., Sullivan (2015)). The solution to these deterministic equations contains the information on the statistical moments of the evolution of the stochastic system.

We use the PC expansion of the stochastic open loop system to compute the feedback gains of a stochastic control law which maximizes the ROA of the closed loop stochastic

system. The option of imposing input constraints on the feedback law is derived for the PC expanded representation and explicitly included in the approach. As the PC expanded system is deterministic, Lyapunov stability methods for deterministic systems can be employed to formulate the conditions on the ROA. The approach builds on results presented in Ahbe et al. (2019). Therein, the connection between the notions of moment stability of a stochastic system and the asymptotic stability of the associated PC expansion were derived. Further, it was shown how an estimate of the ROA of the stochastic system can be recovered from an estimate of the ROA of the PC expanded system. In order to compute the feedback gains and the corresponding inner estimate of the ROA from the Lyapunov stability conditions on the PC expanded system, the conditions and constraints are set into an optimization program. This optimization program is built on results from real algebra (Stengle (1974)) and sum-of-squares programming techniques that were previously employed for ROA computations of deterministic systems. An algorithmic outline of the implementation of the optimization program is provided and demonstrated by the application to two numerical examples. To benchmark the presented control design we compare the results to a linear control design proposed in Fisher and Bhattacharya (2008).

1.1 Notation

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra of the subsets in Ω and μ is a non-negative probability measure on $(\Omega, \mathcal{F})$. A random variable with finite second moment is denoted by $\xi : \Omega \to \mathcal{K} \subseteq \mathbb{R}$, $\xi \in \mathcal{L}_2(\Omega, \mu)$ where $\mathcal{L}_l$, $1 \leq l \leq \infty$, refers to the Lebesgue space (see, e.g. Sullivan (2015) for definitions). A probability distribution λ with mean ν and variance σ^2 is denoted by $\lambda(\nu, \sigma^2)$. The symbol $\sim$ denotes an element with distribution λ. Let $\mathcal{P}^n$ denote the ring of all n-variate polynomials with real coefficients and let $\mathcal{P}^n_{\leq r}$ denote those polynomials of total degree at most $r \in \mathbb{N}_0$. A polynomial $g(x) : \mathbb{R}^n \to \mathbb{R}$, $g(x) \in \mathcal{P}^n_{\leq r}$, is called a sum-of-squares (SOS) if it can be written as $g(x) = \sum_i q_i(x)^2$, $q_i(x) \in \mathcal{P}^n_{\leq r/2}$. Moreover, g is SOS if and only if there is a matrix $Q \succeq 0$ such that $g(x) = v(x)^T Q v(x)$, where $v(x)$ is a vector of monomials, and Q called the Gram matrix. The set of all SOS polynomials in x is indicated by $\Sigma[x]$. The degree of g is denoted by $\partial(g)$.

2. PROBLEM STATEMENT AND BACKGROUND

This work focuses on a feedback control design for stochastic systems with the objective of maximizing the ROA of the closed loop system. We consider continuous time second order random processes with affine input,

$$\dot{x}(t, \xi) = f(x(t, \xi), a(\xi)) + g(x(t, \xi), a(\xi))u(t, \xi), \quad (1)$$

where $x \in \mathbb{R}^n$ is the random state variable, $u \in \mathbb{R}^m$ is a random input variable, $a \in \mathcal{L}_2(\Omega, \mu; \mathbb{R}^l)$ is a random variable representing the parametric uncertainty, and $f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n \times m}$ are polynomial functions in x and a. We assume ξ to have finite support. This is the case in most practical applications. Infinite-support distributions, such as Gaussian distributions, can be limited to a finite support with negligible approximation error (Hover and Triantafyllou (2006)). The initial state of (1) is considered random with $x(t = 0) = x_{\text{ini}}(\xi)$. The control law is based on the state feedback policy,

$$u(t, \xi) = Kh(x(t, \xi)), \quad (2)$$

where $h(x(t, \xi)) : \mathbb{R}^n \to \mathbb{R}^k$ is a vector with entries consisting of polynomials of the components of $x(t, \xi)$, and $K \in \mathbb{R}^{m \times k}$ is the feedback gain matrix. The objective of the feedback control is to stabilize the system around an equilibrium point x_{EP}, whose location is, for conciseness of demonstration [1], assumed to be independent of the uncertainty affecting the system, i.e. $f(x_{\text{EP}}, a) = 0$. We further assume, without loss of generality, the equilibrium point to be the zero point, $x_{\text{EP}} = 0$. Let the closed loop system be denoted by

$$\dot{x}_{\text{cl}} = f_{\text{cl}} \equiv f(x(t, \xi), a(\xi)) + g(x(t, \xi), a(\xi))Kh(x(t, \xi)), \quad (3)$$

and let $\psi_{\text{cl}}(t, \xi, x_{\text{ini}}(\xi))$ denote the solution of (3) at time t with initial condition $x_{\text{ini}}(\xi)$. We then define the region of attraction of the closed loop system (3) as the set of initial conditions, $\mathcal{R}^*$, for which holds

$$\mathcal{R}^* = \{x_{\text{ini}} \in \mathbb{R}^n \,|\, \mathbb{P}\,[\lim_{t \to \infty} d(\psi_{\text{cl}}(t, \xi, x_{\text{ini}}), x_{\text{EP}}) = 0] = 1\}, \quad (4)$$

where d indicates the distance measured in, e.g., the Euclidean norm.

2.1 Polynomial Chaos Expansion

Stochastic processes with finite second moment can be approximated through Polynomial Chaos (PC) expansion. This results in the benefit of a deterministic representation of the system at the cost of a higher state dimension. For an overview see, e.g., Sullivan (2015).

PC expansions use an orthogonal polynomial basis $\mathcal{Q} = \{\Phi_i | i \in \mathbb{N}\} \subseteq \mathcal{P}$ to approximate the random variables or processes. The orthogonal basis satisfies the property

$$\langle \Phi_i(\xi), \Phi_j(\xi) \rangle = \int_\Omega \Phi_i(\xi)\Phi_j(\xi) d\mu(\xi) = \gamma_i \delta_{ij}, \quad (5)$$

where $\gamma_i := \langle \Phi_i(\xi), \Phi_i(\xi) \rangle$ is the normalization factor and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{L}_2$, representing integration (i.e. expectation) with respect to μ. For optimal convergence of the approximation, the orthogonal polynomial basis is chosen such that the orthogonality weighting function corresponds to the type of probability distribution of the random variable.

The PC expansion of a square-integrable real vector-valued random variable $y(\xi) \in \mathcal{L}_2(\Omega, \mu)$ is then

$$y(\xi) = \sum_{i=0}^{\infty} \bar{y}_i \Phi_i(\xi), \quad (6)$$

with vector valued PC coefficients $\bar{y}_i = [\bar{y}_{1_i}, ..., \bar{y}_{n_i}]^T$ which can, for example, be obtained from a Galerkin projection,

$$\bar{y}_i = \gamma_i^{-1} \langle y(\xi), \Phi_i(\xi) \rangle, \quad (7)$$

The expansion series in (6) is infinite and needs to be truncated for practical purposes. Due to the $\mathcal{L}_2$-convergence of the series the approximation error is in general sufficiently small for low orders of the truncation order p (Xiu and Karniadakis (2002)). For the remainder of the paper it is

[1] The proposed approach can be extended to uncertainty dependent equilibrium points which requires the consideration of boundedness as a stability notion. While the extension can be directly derived from results presented in Ahbe et al. (2019), the details go beyond the scope of this paper.

assumed that the truncated series represents the properties under consideration (i.e., stability) of the real system accurately.

In the PC framework the moments of a random variable can be retrieved from the PC coefficients. Consider the vector-valued random variable $y(\xi) \in \mathcal{L}_2(\Omega, \mu)$. With the notation in (6), the first moment, i.e. the mean of $y(\xi)$, is found as

$$\mathbb{E}[y(\xi)] = \langle y(\xi), \Phi_0 \rangle = \bar{y}_0. \tag{8}$$

The second central moment (variance) of $x(t, \xi)$ follows as

$$\sigma^2 := \mathbb{E}[|y(\xi) - \mathbb{E}[y(\xi)]|^2] = \sum_{j=1}^{p} \bar{y}_j^2 \gamma_j, \tag{9}$$

where the sum is to be taken separately over all PC expansion coefficients of each component of y.

The notation for the PC coefficients of a random variable $y \in \mathbb{R}^n$, is

$$\bar{y}_0 := [\bar{y}_{1_0}, \ldots, \bar{y}_{n_0}]^T \in \mathbb{R}^n, \tag{10}$$

$$\bar{y}_J := [\bar{y}_{1_1}, \ldots, \bar{y}_{n_1}, \ldots, \bar{y}_{1_p}, \ldots, \bar{y}_{n_p}]^T \in \mathbb{R}^{n \cdot p}, \tag{11}$$

where we call $\bar{y}_0$ the *mean modes*, and $\bar{y}_J$ the *variance modes*. Together, they present the *stochastic modes*, denoted by

$$\bar{y} := \begin{bmatrix} \bar{y}_0 \\ \bar{y}_J \end{bmatrix}. \tag{12}$$

The PC expansion represents a n-dimensional stochastic system as a $n \cdot (p+1)$-dimensional deterministic system. The deterministic equations are obtained by projecting the truncated series onto each of the $(p+1)$ basis functions. For a demonstration of the expansion see, e.g. Ahbe et al. (2019). The expansion of (1) is denoted by

$$\dot{\bar{x}} := \bar{f}(\bar{x}) + \bar{g}(\bar{x})\bar{u}, \tag{13}$$

where $\bar{x} \in \mathbb{R}^{n(p+1)}, \bar{u} \in \mathbb{R}^{m(p+1)}$ are the vector of PC expansion coefficients of state x and input u, and $\bar{f} : \mathbb{R}^{n(p+1)} \to \mathbb{R}^{n(p+1)}$, $\bar{g} : \mathbb{R}^{n(p+1)} \to \mathbb{R}^{n(p+1) \times m(p+1)}$ the PC coefficient dynamics. The overbar notation indicates variables in the PC expanded representation.

2.2 Stability connection between PC and stochastic system

As shown in Ahbe et al. (2019) an estimate of the ROA of the equilibrium point x_{EP} of a stochastic system can be obtained from an estimate of the ROA of the equilibrium point $\bar{x}_{\text{EP}}$ of the PC expanded system. The objective of the control design is to obtain a control law maximizing the ROA of the stochastic system and thus will be achieved by aiming at maximizing the ROA of the PC expanded system. In the following we briefly summarize results from Ahbe et al. (2019) on the connection of the ROAs of the two system representations and the criteria for certifying a ROA estimate. First, we note that for an uncertainty-independent equilibrium point, $x_{\text{EP}} = 0 \Rightarrow \bar{x}_{\text{EP}} = 0$. An estimate of the ROA is then obtained by the following result.

Theorem 1. Let $\bar{D} \subset \mathbb{R}^{n \cdot (p+1)}$ be a compact domain containing $\bar{x}_{\text{EP}}$. If there exists a continuously differentiable function $V(\bar{x}) : \bar{D} \to \mathbb{R}$ such that

$$V(\bar{x}) > 0 \quad \forall \bar{x} \in \bar{D} \backslash \{\bar{x}_{\text{EP}}\}, \qquad V(\bar{x}_{\text{EP}}) = 0, \tag{14}$$

$$\dot{V}(\bar{x}) = \frac{\partial V(\bar{x})}{\partial \bar{x}} \bar{f}_{cl}(\bar{x}) < 0 \quad \forall \bar{x} \in \bar{D} \backslash \{\bar{x}_{\text{EP}}\}, \tag{15}$$

then $\bar{x}_{\text{EP}}$ is asymptotically stable and $V(\bar{x})$ is a Lyapunov function of the system (13).

The proof follows from standard Lyapunov arguments. If these conditions are satisfied for all $\bar{x}$ in a sublevel set

$$\bar{\mathcal{R}} = \{\bar{x} \in \bar{D} \,|\, V(\bar{x}) \leq \rho\}, \tag{16}$$

where ρ is a positive scalar and $\bar{\mathcal{R}} \subseteq \bar{D}$, then $\bar{\mathcal{R}}$ is an inner estimate of the region of attraction of $\bar{x}_{\text{EP}}$. The connection between the moment stability of a stochastic system and the asymptotic stability of its PC expansion is then used to conclude that, given an $\bar{\mathcal{R}}$ estimate, the associated set $\mathcal{R} = \{x_{\text{ini}} \in \mathbb{R}^n \,|\, x_{\text{ini}}(\xi) \sim \lambda(\bar{x}_0, \sigma^2(\bar{x}_J), ..), \forall \bar{x} \in \bar{\mathcal{R}}\}$ is an inner estimate of $\mathcal{R}^*$, the ROA of the stochastic system. These results enable the estimation of the ROA of a stochastic system by using stability analysis tools for deterministic systems. They represent the starting point for the main technical result of the paper presented in the next Section.

3. FEEDBACK CONTROL DESIGN

In this work, the stochastic controller (2) is considered with the aim of obtaining a state feedback law maximizing the ROA. The approach offers flexibility in choosing the explicit expression of the state function $h(x(t, \xi))$.

3.1 Stochastic state feedback law

We use the stochastic state feedback law (2) and focus on the design of the gain matrix K as well as the state vector $h(x(t, \xi))$. A linear version of this feedback law has been used in Fisher and Bhattacharya (2009) for the design of a Linear Quadratic Regulator (LQR). Since in this work we are dealing with nonlinear systems, the control law considered here contains polynomials in the state as feedback variables. As the control design proposed here considers the PC expansion of the closed loop system, the control law (2) needs to be expanded. Note that by considering the PC expanded system in the control design task, the stochastic information on x is directly exploited in the computation of K. Expanding the control law (2) in the PC framework as in equation (6) results in

$$\bar{u}_{i_j} = \gamma_j^{-1} \langle K_i h(x), \Phi_j(\xi) \rangle, \tag{17}$$

where $i = 1, \ldots, m$, $j = 0, \ldots, p$ and K_i is the i-th row vector of K. Note that the dimension of K depends only on the dimension of the stochastic input and the stochastic vector $h(x)$, and is independent on the truncation order p of the PC expansion.

Remark 3.1. There are other possibilities for feedback laws, e.g. where the input is considered deterministic or where K is a random variable (see, e.g. Fisher and Bhattacharya (2009)). The approach proposed here can be used for other feedback laws as well, however in applications these laws require knowledge of the current probability density function of the state vector. While for linear systems there exist well-established state estimation techniques providing the probability density of the state, estimates of the probability density are harder to obtain for uncertain nonlinear systems. Thus, we limit our focus to the stochastic state feedback law.

Remark 3.2. In this work we are assuming that x_{EP} is a locally asymptotically stable equilibrium point and the

control design only aims at maximizing the ROA. The approach also remains valid in principle if x_{EP} is unstable. This is done by first stabilizing x_{EP} with standard techniques (e.g. feedback linearization) and then applying the design scheme proposed here to increase the ROA of the stabilized system.

3.2 Input constraints

For the stochastic control law, input limits can be imposed with the aim of obtaining a controller which maximizes the certified ROA while respecting the system's physical constraints. As the analysis deals with *PC expanded* systems, the constraints need to be expressed in terms of PC coefficients. Let the constraints on the stochastic input be

$$u^L \leq u(t,\xi) \leq u^U. \tag{18}$$

Due to the stochastic nature of u, the constraints are expressed in terms of the statistical properties of the input, which are provided by the PC expansion. More precisely, we consider the mean with the addition of one standard deviation of the control input, given by equations (8) and (9) applied to the stochastic signal u, and constrain these to remain within specified limits. This results in

$$u^L \leq \bar{u}_0 - \sigma, \qquad u^U \geq \bar{u}_0 + \sigma, \tag{19}$$

with

$$\sigma_i := \left(\sum_{j=1}^{p} \bar{u}_{i_j}^2 \gamma_j \right)^{\frac{1}{2}}, \tag{20}$$

for each component $i = 1, .., m$ in of the input u. Note that due to the square root, equation (19) does not result in a polynomial expression. As the design method proposed here hinges on both the constraints to be in polynomial form and the matrix K to only appear linearly, we introduce the following relaxation of (19). Since γ_j is positive by definition, each term in the summation in (20) is positive and thus the following holds

$$\left(\sum_{j=1}^{p} \bar{u}_j^2 \gamma_j \right)^{\frac{1}{2}} \leq \sum_{j=1}^{p} |\bar{u}_j| \gamma_j^{\frac{1}{2}}. \tag{21}$$

The right-hand side of (21) provides an upper bound on the standard deviation of u. By considering the maximum negative and maximum positive realizations of $\bar{u}_j$ separately, the right-hand side of (21) can be expressed as two polynomial constraints, which will be explained in more detail in the following section.

Remark 3.3. Note that if the uncertainty distribution has finite support (as assumed here and is usually the case in practice) then the constraints in (19) impose hard constraints on the input. In the more general case of uncertainty distribution with infinite support the constraint violations cannot be excluded due to the tails of the distributions. In that case the constraints as formulated here would have primarily the effect of penalizing the input magnitude.

4. ALGORITHM FOR COMPUTING K

In this section we show how the stochastic control law (2) is computed such that the ROA of the closed loop system is maximized. Outlines of the algorithmic implementation of the computations are provided for both unconstrained and constrained input cases.

4.1 Maximizing ROA over K

Leveraging the stability criteria stated in Theorem 1, the following nonlinear optimization problem can be formulated for the computation of the matrix K and concurrent maximization of the ROA estimate $\bar{\mathcal{R}}$.

$$\max_{V(\bar{x}),K} \quad \text{vol}(\bar{\mathcal{R}}(\bar{x})) \tag{22a}$$

$$\text{subject to} \quad V(\bar{x}) > 0,\, V(0) = 0, \tag{22b}$$

$$\dot{V}_{\text{cl}} = \frac{\partial V(\bar{x})}{\partial \bar{x}} \bar{f}_{\text{cl}}(\bar{x}, K) < 0, \tag{22c}$$

$$u^L \leq \bar{u}(\bar{x}, K) \leq u^U, \tag{22d}$$

where we use $V = v(\bar{x})^T Q v(\bar{x})$ with $v(\bar{x})$ being the vector of monomials in $\bar{x}$ up to a chosen degree, and $\bar{\mathcal{R}}$ as defined in (16). Constraint (22d) is only present if there are input constraints.

4.2 Algorithmic implementation

If all equations in (22) are polynomial, the optimization program can be solved using results from real algebraic geometry, in particular the Positivstellensatz as stated in Stengle (1974). This provides a tool to formulate the conditions in (22) as semi-algebraic set emptiness conditions. These can then be relaxed to sum-of-squares (SOS) programs. If the SOS program can be formulated such that the objective and constraints are convex in the decision variables then it can be solved via semidefinite programming (SDP), see, e.g., Parrilo (2000) for more details.
Thus, in order to efficiently implement the optimization problem (22) via SOS programs, the objective function, and therefore the measure of the volume of $\bar{\mathcal{R}}$, need to be convex. For a positive quadratic form, the geometric mean of the eigenvalues of the Gram matrix is a convex expression and an inversely monotonic function of the sublevel set volume. As higher order V have the potential to verify larger estimates of $\bar{\mathcal{R}}$, a surrogate set is used in order to have a convex measure and thus maximize the sublevel set volume efficiently (see, e.g., Topcu and Packard (2009); Ahbe et al. (2018)). The surrogate set consists in a quadratic form $b(\bar{x}) := \bar{x}^T B \bar{x}$ with the sublevel set $\mathcal{B} = \{\bar{x} |\, b(\bar{x}) \leq 1\}$. Imposing the constraint $\mathcal{B} \subseteq \bar{\mathcal{R}}$, maximizing over the geometric mean of the eigenvalues of B then leads to a maximization of $\bar{\mathcal{R}}$.
The stochastic input constraint (19) is represented through the upper bound in (21). For the implementation of the absolute value in a polynomial constraint, additional steps are required which are presented in the following. For a computed K, let q_j be the maximum absolute value of the j-th term in the right hand side of (21) over all $\bar{x}$ in the sublevel set ρ of V. Then the input constraint (19) can be written as

$$u^L \leq \bar{u}_0 - \sum_{j=1}^{p} q_j, \quad u^U \geq \bar{u}_0 + \sum_{j=1}^{p} q_j, \tag{23}$$

With the above procedure we obtain the following SOS optimization program, where for clarity of presentation a single input u is considered and the dependence on $\bar{x}$ and K are dropped.

$$\max_{V,K,B,s_i,q_j} \quad -\det(B)^{1/n(p+1)} \tag{24a}$$

subject to

$$V - l \in \Sigma[\bar{x}], \tag{24b}$$
$$-s_1(1-b) - (1-V) \in \Sigma[\bar{x}], \tag{24c}$$
$$-\dot{V}_{\mathrm{cl}} - s_2(1-V) - l \in \Sigma[\bar{x}], \tag{24d}$$
$$-s_3\left(\bar{u}_0 + \sum_{j=1}^{p} q_j - u^U\right) - s_4(1-V) \in \Sigma[\bar{x}], \tag{24e}$$
$$-s_5\left(u^L - \left(\bar{u}_0 - \sum_{j=1}^{p} q_j\right)\right) - s_6(1-V) \in \Sigma[\bar{x}], \tag{24f}$$

for each j :

$$s_7(q_j - \bar{u}_j\gamma_j^{\frac{1}{2}}) - s_8(1-V) \in \Sigma[\bar{x}], \tag{24g}$$
$$s_9(q_j + \bar{u}_j\gamma_j^{\frac{1}{2}}) - s_{10}(1-V) \in \Sigma[\bar{x}], \tag{24h}$$
$$s_{i,i=1,\ldots(6+4p)} \in \Sigma[\bar{x}], \tag{24i}$$

where the multipliers $s_i(\bar{x}), i = 1,..,(6+4p)$ are SOS polynomials in $\bar{x}$ which result from the Positivstellensatz and certify the solution of the program to adhere to the constraints. Also resulting from the Positivstellensatz are the terms $l(\bar{x}) = \epsilon\bar{x}^T\bar{x}$, $\epsilon \ll 1$, in (24b) and (24d) which guarantee that $\bar{x} = 0$ is not included in the constrained set. Constraint (24c) ensures the containment of the surrogate set in the sublevel set of the Lyapunov function. The sublevel set size ρ has thereby been fixed to 1 as optimization over ρ is redundant with optimizing over Q. The constraints (24e)-(24h) enforce the input constraints on the computation of K and $\bar{\mathcal{R}}$. Note that for each PC coefficient of u there are two additional constraints (24g)-(24h), leading to four additional multipliers and thus making the total amount of multipliers dependent on the truncation order p. In the implementation, the variables q_j are 'measures' of the maximum absolute values of both the positive and negative values of $\bar{u}_j\gamma_j^{\frac{1}{2}}$ over all $\bar{x}$ in $\bar{\mathcal{R}}$ and add, respectively subtract, them from the mean value $\bar{u}_0$. Constraints (24e)-(24f) then ensure (23). In the case of no input constraints the optimization program only includes (24a)-(24d) and (24i).

Due to bilinearly appearing decision variables in the constraints, the SOS program (24) can still not directly be solved as an SDP. To circumvent the bilinearities and obtain convex constraints, a potentially suboptimal iterative scheme is proposed consisting of an iterative loop over three steps. A pseudocode of the iterative scheme is shown in Algorithm 1. The program returns both the matrix K as well as the corresponding ROA estimate $\bar{\mathcal{R}}$. The algorithm is initialized by finding a suitable Lyapunov function, e.g. from the linearisation of the system around $\bar{x}_{\mathrm{EP}}$ with $K = 0$, solving the Lyapunov matrix inequality, and scaling the result appropriately. An initial sublevel set size is then simply obtained by choosing an initial diagonal B-matrix with entries sufficiently large. The initial K can be taken as a matrix with small nonzero or with zero entries. The iteration over the three steps is then concerned with maximizing the surrogate set, i.e. the ROA estimate, while searching for appropriate K values. In particular, Step 1 consists in finding multipliers for the current Q, B and K. In Step 2, K is optimized for and in Step 3 the surrogate set volume is maximized over Q with the multipliers and K kept fixed. Note, that in the case of no constraints on the input, constraints (24e)-(24i) including the variables q_j are omitted from Steps 1-3. The iteration terminates when a predefined convergence criteria on the size of the surrogate set ($\mathrm{convCrit}_B$) is reached.

Algorithm 1 Find control to maximize ROA

Input: p, $\partial(s_i)$, $\partial(V)$, $\mathrm{convCrit}_B$, $h(\cdot)$, u^L/u^U
Output: K ,V
procedure MAXROAESTIMATE
  $\bar{f}(\bar{x}, K) \xleftarrow{\mathrm{PCE}} f(x, a, u)$
  Initialization:
  set $K = 0$
  $Q_{\mathrm{ini}} \leftarrow$ Lyapunov inequality for $\left.\frac{\partial \bar{f}(\bar{x})}{\partial \bar{x}}\right|_{\bar{x}_{\mathrm{EP}}}$
  choose B small enough such that (24c) is feasible
  $K_{\mathrm{ini}} \leftarrow <<1$ (or zero)
  Iteration:
  $k \leftarrow 0$
  repeat
    $k \leftarrow k + 1$
    Step 1: $s_i \leftarrow$ fix Q, B, K, q_j, solve (24c)-(24i)
    Step 2: K, $q_j \leftarrow$ fix Q, s_i, solve (24d)-(24f)
    Step 3: Q, $B \leftarrow$ fix s_i, K, q_j, solve (24)
  until $\det(B)_{k-1} - \det(B)_k < \mathrm{convCrit}_B$
end procedure

4.3 Recovering $\mathcal{R}$ from $\bar{\mathcal{R}}$

Algorithm (1) returns an estimate of $\bar{\mathcal{R}}$ that describes a set in terms of the PC coefficients $\bar{x}$. In order to obtain from $\bar{\mathcal{R}}$ an inner estimate $\mathcal{R}$ of the true ROA of the stochastic system, $\mathcal{R}^*$, we use the optimization program proposed in Ahbe et al. (2019). This program computes $\mathcal{R}$ from the $\bar{\mathcal{R}}$ estimate by specifying the stochastic properties of the initial conditions. For example, by fixing the variance of the initial state to a desired value, the set $\mathcal{R}$ is obtained in terms of the mean of the initial state. In the following examples we set the variance of the initial state to zero. This gives the computation of $\mathcal{R}$ directly from the Lyapunov sublevel set representing $\bar{\mathcal{R}}$ by setting all PC coefficients of the variance modes in $\bar{\mathcal{R}}$ to zero.

5. ILLUSTRATIVE EXAMPLES

We demonstrate the proposed control design on two examples. Both stochastic systems are open loop stable and a feedback control will be used to enlarge the ROA. In order to benchmark our approach, we compare it, firstly, with the ROA estimate computed for the open loop system. The open loop ROA estimate is thereby obtained from applying the ROA computations outlined in Ahbe et al. (2019). Secondly, we compare our approach with one of the few available PC expansion based control algorithms. This control algorithm consists in LQR control design proposed in Fisher and Bhattacharya (2009) for PC expanded *linear* stochastic systems. The example dynamics presented here are therefore linearized around their equilibrium point and the feedback law "stochastic state feedback with constant deterministic gain" in Fisher and Bhattacharya (2009), Sec. 5.2.2, is applied. The control design proposed therein results in bilinear matrix inequalities (BMI). This is solved here using PENLAB (Fiala et al. (2013)) which returns

the LQR values for the gain matrix. In order to obtain the ROA of the LQR controlled system, the open loop ROA computations are applied to the LQR controlled closed loop system.

5.1 2D stochastic dynamics

The first example considers system dynamics from Chesi (2004), p. 146, with modifications to introduce uncertainty,

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - c(\xi)(x_1^2 + x_2^3) + x_1 u,\\ \dot{x}_2 &= -2x_2 - c(\xi)x_1^2 + u,\end{aligned} \tag{25}$$

where $c(\xi)$ is a random variable coming from a uniform distribution, $c(\xi) \sim \text{Unif}(0.8, 1.2)$ with $\xi \sim \text{Unif}(-1, 1)$. The Legendre polynomials, associated with uniform distributions, are used in the PC expansion. The closed loop system is analysed and compared for two choices of h which differ in the degree of the polynomial entries, $h_1 = [x_1, x_2]^T$ with $K_1 \in \mathbb{R}^2$, and $h_2 = [x_1, x_2, x_1^2, x_1x_2, x_2^2]^T$ with $K_2 \in \mathbb{R}^5$. In this example the input is unconstrained. In order to decide on a truncation order of the PC expansion we simulate the evolution of the stochastic modes of the system for a large p starting from various initial conditions in the region of interest and then set p such that it captures the significant modes. Here, $p = 2$, i.e. there are three significant modes for each state dimension, which results in a six-dimensional deterministic system.
Figure 1 shows the ROA estimates $\mathcal{R}$ for both open loop and closed loop dynamics obtained using quadratic and quartic Lyapunov functions. Additionally, the $\mathcal{R}$ estimate for the LQR-controlled system is shown. The results show how the proposed feedback design increases the estimated $\mathcal{R}$ of the stochastic system. In particular, the results show how higher order polynomials in $h(x)$ can lead to larger increases of $\mathcal{R}$ and how higher degree Lyapunov functions can certify larger estimates. It is stressed that the improvements on the ROA are mainly due to the fact that a nonlinear control design formulation is employed, rather than due to the adoption of a polynomial basis for h. This is shown by the case of h_1 which is a linear controller and significantly outperforms the LQR. The $\mathcal{R}$ estimate for the LQR controlled system found for this example shows an improvement over the $\mathcal{R}$ estimate for the open loop system. It is, however, small compared to the nonlinear controllers. Note that since the BMI has, in general, local optima and the obtained solution depends on the initialization, the LQR-$\mathcal{R}$ results shown here are not unique results.

5.2 Short period aircraft dynamics

The second example consists of the 2-D aircraft short period dynamics from Chakraborty et al. (2011). While in this reference the dynamics are nominal, two uncertain parameters affecting the nonlinear part of the system are considered here. With x_1 representing the angle of attack (in radians) and x_2 the pitch rate (in radians/seconds), the dynamics are given by

$$\begin{aligned}\dot{x}_1 &= c_1(\xi)(-1.492x_1^3 + 4.239x_1^2 + 0.003x_1x_2 + 0.006x_2^2) -\\ &\quad - 3.236x_1 + 0.923x_2 + (-0.317 + 0.240x_1)u,\\ \dot{x}_2 &= c_2(\xi)(-7.228x_1^3 + 18.36x_1^2 + 1.103x_2^3) -\\ &\quad - 45.34x_1 - 4.372x_2 + (-59.99 + 41.5x_1)u,\end{aligned} \tag{26}$$

where $c_1(\xi) \sim \text{Unif}(0, 2)$ and $c_2(\xi) \sim \text{Unif}(0.5, 1.5)$, with $\xi \sim \text{Unif}(-1, 1)$, are random variables from a uniform distribution. The input u represents the elevator deflection (in radians). A truncation order of $p = 2$ is found to capture the significant modes. Further, the vector $h = [x_1, x_2]^T$ is chosen for the control law (2). The dynamics (26) in their nominal form with $c_1 = 1$, $c_2 = 1$, are open loop stable. To investigate the effects of uncertainty and feedback control on the stability of the system, first the ROA for both the nominal open loop dynamics, OL$_n$-$\mathcal{R}$, and the stochastic open loop dynamics, OL$_s$-$\mathcal{R}$, were computed. Then, Algorithm 1 was used to compute the ROA, CL-$\mathcal{R}$, and the corresponding control law for the stochastic closed loop system without input constraints. Figure 2 shows the ROA estimates for each case, and additionally a ROA estimate for the LQR-controlled system. Compared to the set OL$_n$-$\mathcal{R}$, the estimate OL$_s$-$\mathcal{R}$ is found to be significantly smaller. The results for CL-$\mathcal{R}$ reveal that the controller is able to stabilize the stochastic closed loop system such that the feedback not only counteracts the uncertainty but further enlarges the ROA. In this example this is also found for the ROA estimate of the LQR-controlled system, LQR-$\mathcal{R}$. For illustration, sample trajectories for various realizations of the uncertain parameters over the distribution range are shown for both the open and closed loop system starting from an initial state which is inside of CL-$\mathcal{R}$ but outside of OL$_s$-$\mathcal{R}$. In accordance with the ROAs, the open loop trajectories diverge while the closed loop trajectories converge to the equilibrium point.
In the left plot of Figure 3, the evolution of the closed loop system states and input for an initial condition inside CL-$\mathcal{R}$ and evaluated for different uncertainties covering the whole distribution range is shown. The plot reveals input magnitudes exceeding by far the physical limits of ± 0.5 rad for the elevator deflection. We thus impose the constraints $-0.5 \text{ rad} \leq u(\xi) \leq 0.5 \text{ rad}$ as explained in Section 3.2, and recalculate the $\mathcal{R}$ estimate and resulting feedback gains from Algorithm 1. The input constrained CL-$\mathcal{R}$, shown with the purple line in Figure 2, shows how the ROA shrinks for the constrained case relative to the unconstrained. The right plot in Figure 3 shows how the constrained input for the same initial condition and range of uncertainties as for the unconstrained input now

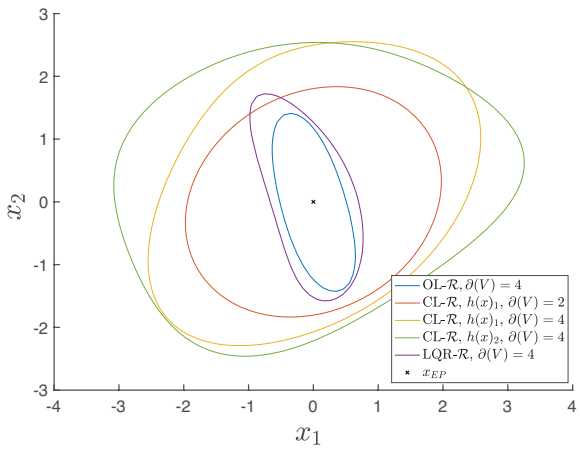

Fig. 1. Resulting ROA estimates for the open loop (OL-$\mathcal{R}$), closed loop (CL-$\mathcal{R}$), and LQR-controlled (LQR-$\mathcal{R}$) stochastic system (25) using, for comparison, quadratic and quartic Lyapunov functions and two different state feedback vectors $h(x)_1$ and $h(x)_2$.

remains within the prescribed bounds for all realizations of the uncertainty over the given range.

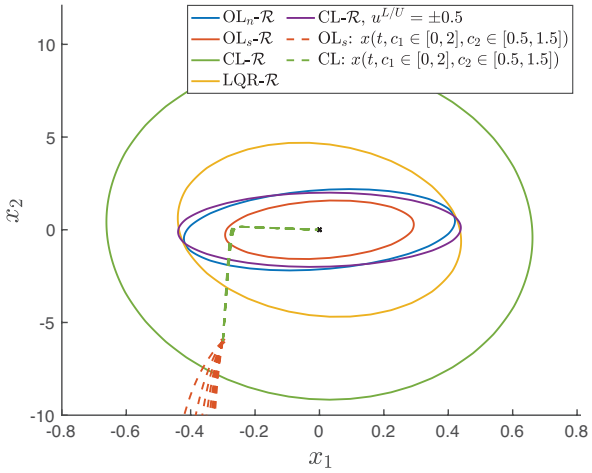

Fig. 2. Resulting $\mathcal{R}$ estimates for the nominal open loop, stochastic open loop and stochastic closed loop system for both without and with input constraints, using quadratic Lyapunov functions in Algorithm 1. For the deterministic initial condition $x_{\text{ini}} = [-0.3; 6]$ and for various realizations over the full range of c_1 and c_2, sample trajectories of the stochastic open loop system, found to be all diverging, and for the closed loop system, found to be all converging, are shown.

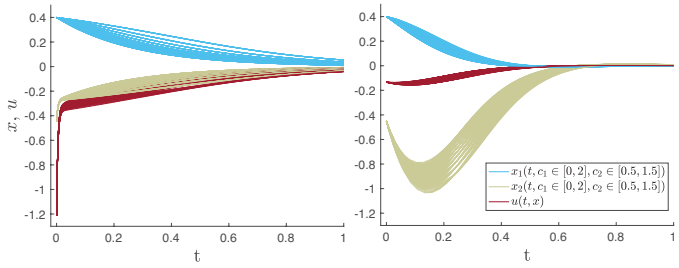

Fig. 3. *Left plot:* Closed loop system trajectories of the states and unconstrained input starting from the deterministic initial condition $x_{\text{ini}} = [0.4; -0.45]$ and for 10 different uncertainty realizations of each uncertain parameter c_1, c_2 covering the whole range of both distributions (resulting in each 100 trajectories).
Right plot: Same simulation configuration as in left plot, but here the input was computed under the constraints $u^L = -0.5$ rad, $u^U = 0.5$ rad.

6. CONCLUSION

We propose a method to obtain feedback gains which maximize an inner estimate of the ROA of a stochastic closed loop system. To this end, the Polynomial Chaos expansion framework is employed to represent the stochastic equations by higher dimensional deterministic ones. The control design is based on Lyapunov stability for deterministic systems where the resulting stability conditions are verified via sum-of-squares programs. We demonstrate by two examples the various features of the control design and the corresponding ROAs which result from the proposed method. While the computational implementation of the control design is still limited for larger systems by current SOS programming capabilities, the proposed approach offers flexibility in choosing the stochastic feedback law and imposing input constraints. Future work could aim at reducing the conservatism of the constraint satisfaction and including controller performance measures.

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