Estimating the region of attraction of uncertain systems with invariant sets ⋆

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Abstract: In this article the problem of estimating the Region of Attraction (ROA) for polynomial nonlinear systems subject to modeling uncertainties is studied. Based on recent theoretical studies on the calculation of positively invariant sets, this article proposes an optimization problem which allows robust inner Estimates of the Region of Attraction (rERA) to be evaluated. The uncertainties, which can generically be time-invariant or time-varying, are described as semialgebraic sets, and the problem is solved numerically by means of Sum Of Squares relaxations, which allow set containment conditions to be enforced. The ensuing optimization entails non-convex constraints, and an iterative algorithm to enlarge the provable invariant level set is discussed. The proposed algorithm is applied to two study cases of increasing complexity. Further, in order to benchmark the proposed rERA algorithm, comparisons are shown with a class of well established algorithms based on Lyapunov functions level sets. The results showcase the prowess of the proposed approach and its advantages in terms of accuracy and computational time, particularly as the size of the system increases.

Keywords: Region of attraction, Uncertain systems, Local analysis, Sum Of Squares

1. INTRODUCTION

The Region of Attraction (ROA) associated to an equilibrium point x^* of a nonlinear system is the set of all the initial conditions from which the trajectories of the system converge to x^* as time goes to infinity (Khalil, 1996), and its knowledge is of practical interest to guarantee the safe operation of nonlinear systems.
Finding the exact region of attraction analytically might be difficult and several algorithms have been proposed to calculate inner Estimates of the Region of Attraction (ERA), which can be broadly classified into two categories: Lyapunov methods and non-Lyapunov methods.
The former build on the invariance and contractiveness properties held by Lyapunov functions (LF) sublevel sets. A common approach for finding a suitable (of generic degree) LF for polynomial nonlinear systems is to use Sum of Squares (SOS) techniques and recast the problem as a set of SemiDefinite Programs (SDPs) (Chesi, 2011).
Non-Lyapunov methods have also been studied to reduce the conservatism associated with the aforementioned approaches. For example, in a recently published work (Valmorbida and Anderson, 2017), the recipes for calculating ERA are expressed in terms of positively invariant sets. This approach, prompted by LaSalle's theorem (Khalil, 1996), still uses Lyapunov stability concepts but prescribe weaker conditions for the function used to define the ERA.

When a more realistic description for the plant's dynamics is considered, the study of local stability should take into account the presence of uncertainties. In (Topcu and Packard, 2009) an algorithm restricted to systems with a specific dependence on the uncertainties (e.g. uncertain parameters appearing affinely) was proposed, based on parameter-independent LF, i.e. a single Lyapunov function is used to certify the local stability of a system over its entire uncertainty set. This was refined in (Topcu et al., 2008) allowing for a branch-and-bound improvement to alleviate the conservatism associated to the parameter-independent LF. Other studies considered parameter-dependent LFs (Chesi, 2004; Tan and Packard, 2008), with the ensuing SOS-based optimization problem featuring a substantial increase in computational burden.

The main contribution of the work is to propose an algorithm, building on the result for the nominal case of (Valmorbida and Anderson, 2017), to determine robust inner Estimates of the Region of Attraction (rERA), i.e. ERA of systems with uncertainties. In view of the well recognised and conflicting aspects mentioned before, the rERA is expressed as a parameter-independent invariant set, but a parameter-dependent LF is involved in its calculation. Even though the result is general, the proposed implementation is based on Sum of Squares relaxations of the set containment conditions defining the rERA. The characterization of the uncertainties is via semialgebraic sets, and thus their nature and the vector field's dependence on them are relatively generic.
Two case studies from the literature, featuring increasing complexity, are used to validate the proposed approach and to quantify its performance by comparing the results with those from the LF-based algorithms of references (Topcu and Packard, 2009; Topcu et al., 2008).

The layout of the article is as follows. Section 2 provides a cursory introduction to the basics of the work.

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Section 3 presents an approach to compute rERA in the framework of positively invariant sets, and describes an iterative algorithm to numerically solve the problem. This is subsequently applied in Section 4, where results are also discussed. Section 5 finally presents the Conclusions.

2. BACKGROUND

The set of functions $g(x) : \mathbb{R}^n \to \mathbb{R}$ which are m-times continuously differentiable is denoted by $\mathcal{C}^m$. For $x \in \mathbb{R}^n$, the set of all polynomials in n variables is denoted by $\mathbb{R}[x]$. For $g \in \mathbb{R}[x]$, $\partial(g)$ denotes the degree of g. Given a scalar $c > 0$, the level set of g and its boundary are defined as:

$$\begin{aligned} \varepsilon_{g,c} = \varepsilon(g,c) &:= \{x \in \mathbb{R}^n : g(x) \leq c\} \\ \partial\varepsilon(g,c) &:= \{x \in \mathbb{R}^n : g(x) = c\} \end{aligned} \quad (1)$$

A polynomial $g(x)$ is said to be a Sum of Squares (SOS) if there exists a finite set of polynomials $g_1(x), ..., g_k(x)$ such that $g(x) = \sum_{i=1}^{k} g_i^2(x)$. The set of SOS polynomials in x is denoted by $\Sigma[x_1, ..., x_n]$, abbreviated here with $\Sigma[x]$. The importance of SOS polynomials lies on their connection with convex optimization (Parrilo, 2003). Namely, $g \in \Sigma[x]$ if and only if $g = z^T Q z$ and $Q = Q^T \succeq 0$ (i.e. Q is positive semidefinite), where z is a vector gathering the monomials of g. This problem can be recast as a semidefinite program (SDP) and there are freely available software toolboxes to solve this in an efficient manner. In this work, the software SOSOPT from the suite of libraries (Balas et al., 2009) will be used in conjunction with Sedumi (Sturm, 1999).
As for the computational aspects, note that if g is dense (i.e. no sparse monomials), the size of the SDP problem grows polynomially with n if $\partial(g)$ is fixed (and vice versa), but it grows exponentially if both n and $\partial(g)$ increase.

The algorithms employed in this work involve finding functions that satisfy set containment conditions. Known results from real algebraic geometry can be employed to tackle this problem when dealing with polynomial functions. In particular, an application of the Positivstellensatz (P-satz) Theorem allows the following property to be stated.

Lemma 1. (Parrilo, 2003) Given $h, f_0, ..., f_r \in \mathbb{R}[x]$, the following set containment holds

$$\big\{x : h(x) = 0, f_1(x) \geq 0, ..., f_r(x) \geq 0\big\} \subseteq \big\{x : f_0(x) \geq 0\big\} \quad (2)$$

if there exist *multipliers* $p \in \mathbb{R}[x]$, $s_{01}, ..., s_{0r} \in \Sigma[x]$ such that

$$p(x)h(x) - \sum_{i=1}^{r} s_{0i}(x) f_i(x) + f_0(x) \in \Sigma[x] \quad (3)$$

The following result establishing an equivalence between set containments is also recalled:

Lemma 2. (Tan, 2006) For each y satisfying $g_3(y) \leq 0$,

$$\begin{aligned} &\{x \mid g_1(x,y) \leq 0\} \subseteq \{x \mid g_2(x,y) \leq 0\} \qquad \text{iff} \\ &\{(x,y) \mid g_1(x,y) \leq 0, g_3(y) \leq 0\} \subseteq \{(x,y) \mid g_2(x,y) \leq 0\} \end{aligned} \quad (4)$$

The last set containment can be easily enforced with the application of Lemma 1. Indeed with reference to (3) it is enough to take $f_0 = g_2$, $f_1 = g_1$ and $f_3 = g_3$.
In the following, all the set containments will be recast in the form of (2), for which sufficient conditions can be expressed as SOS constraints (3).

Consider an autonomous nonlinear system of the form

$$\dot{x} = f(x), \quad x(0) = x_0 \quad (5)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field. The vector $x^* \in \mathbb{R}^n$ is called a *fixed* or *equilibrium* point of (5) if $f(x^*) = 0$. Let $\phi(t, x_0)$ denote the solution of (5) at time t with initial condition x_0. The ROA associated with x^* is defined as

$$\mathcal{R} := \big\{x_0 \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t, x_0) = x^*\big\} \quad (6)$$

That is, $\mathcal{R}$ is the set of all initial states x_0 that eventually converge to x^*.

Consider now the case when the system is subject to uncertainties:

$$\dot{x} = f(x, \delta), \quad x(0) = x_0, \quad \delta \in \Delta \quad (7)$$

where $f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n$ is the uncertain vector field and Δ denotes a generic bounded uncertainty set. It is assumed, as commonly done in the literature (Chesi, 2004; Topcu and Packard, 2009), that $f(0, \delta) = 0 \; \forall \delta \in \Delta$, i.e. the origin is the equilibrium point regardless of the uncertainties.
The robust Region of Attraction (rROA) is defined as:

$$\mathcal{R}_\delta := \cap_{\delta\in\Delta} \big\{x_0 \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t, x_0, \delta) = 0\big\} \quad (8)$$

where $\phi(t, x_0, \delta)$ is the solution of (7) at time t with initial condition x_0 and subject to $\delta \in \Delta$. That is, $\mathcal{R}_\delta$ is the intersection of the ROAs for all systems governed by (7).

3. COMPUTATION OF ROBUST ERA WITH INVARIANT SETS

In the Introduction it was stated that a more realistic description of the vector field of a nonlinear system entails the inclusion of uncertainties. These may stem from different sources - for example: errors due to modeling assumptions (e.g. a local polynomial approximation of a generic vector field with associated bounded error); parameters with uncertain values; and higher order terms truncation to make the problem numerically tractable (recall the dependence of the SDPs on $\partial(g)$).

In this section, the problem is first theoretically framed into the context of positively invariant sets, and an algorithm to compute robust inner Estimation of Regions of Attraction (rERA) is proposed (Section 3.1). Then, numerical aspects and features of the algorithm are discussed (Section 3.2).

3.1 An algorithm for robust inner estimates of ROA

A standard approach to calculate estimates of ROA for nominal systems (5) consists in applying the Lyapunov's direct method and calculating the largest possible level set of a Lyapunov function (LF), in view of its invariance and contractiveness properties.

As pointed out by LeSalle's theorem (Khalil, 1996), this characterization is usually conservative and the following result, which relaxes the search to positively invariant (but not contractive) level sets of a function R, has been proposed in the literature:

Theorem 1. ((Valmorbida and Anderson, 2017), Th. 1) If there exist $R, V_N : \mathbb{R}^n \to \mathbb{R}$, with $R, V_N \in \mathcal{C}^1$, and a positive scalar γ satisfying:

$$\nabla R(x) f(x) < 0 \qquad \forall x \in \partial\varepsilon(R,\gamma) \tag{9a}$$
$$V_N(0) = 0 \quad \text{and} \quad V_N(x) > 0 \quad \forall x \in \varepsilon(R,\gamma)\backslash 0 \tag{9b}$$
$$\nabla V_N(x) f(x) < 0 \qquad \forall x \in \varepsilon(R,\gamma)\backslash 0 \tag{9c}$$
$$\varepsilon(R,\gamma) \text{is compact and} \quad 0 \in \varepsilon(R,\gamma) \tag{9d}$$

then $x_0 \in \varepsilon_{R,\gamma}$ implies $x_0 \in \mathcal{R}$.

The proof of this result can be found in the reference. The fundamental idea is that $\varepsilon(R,\gamma)$ is a positively invariant set, due to (9a)–(9d), and that all trajectories initiated from it converge to a level set of some LF– which is contractive and invariant because of (9b)–(9c), therefore guaranteeing such set to be an ERA. Note that the function R defining the level set only requires negativity of its gradient on the set boundary.

The case with uncertainties was only marginally considered in (Valmorbida and Anderson, 2017) , and no indications on possible implementations were given.

In this paper it is proposed to describe Δ as a semialgebraic set (Anderson and Papachristodoulou, 2017):

$$\Delta = \left\{\delta \in \mathbb{R}^{n_\delta} : m_i(\delta) \geq 0, i = 1, ..., j\right\} \tag{10}$$

where the functions m_i are polynomials in δ, whose definition will be discussed later. This strategy is quite general and allows both time-invariant and time-varying uncertainties to be taken into account, as well as norm bounded operators. Moreover, no hypotheses on how the uncertainties enter the vector field are made. This is different from other approaches in the literature where, for example, f is required to depend affinely on the uncertain parameters, which are supposed to lie in a given polytope (Topcu and Packard, 2009).

Theorem 1 involves finding functions that satisfy set containment conditions. In order to make the problem computationally tractable, interest is restricted to polynomial vector fields f, and thus, applying Lemma 1, the problem of finding an ERA can be recast as an SOS optimization. The following Lemma allowing to study robust ERA within the framework of invariant sets is then stated:

Lemma 3. Given $R \in \mathbb{R}[x]$, $V_N \in \mathbb{R}[x,\delta]$ with $V_N(0,\cdot) = 0$, and γ a given positive constant. Then, if there exist SOS polynomials $s_1, s_2, s_{0i}, s_{1i}, s_{2i}$ and a polynomial s_0 such that:

$$-\nabla R f - s_0(\gamma - R) - \Gamma_{0j} \quad \in \Sigma[x,\delta] \tag{11a}$$
$$V_N - s_1(\gamma - R) - \Gamma_{1j} \qquad \in \Sigma[x,\delta] \tag{11b}$$
$$-\nabla V_N f - s_2(\gamma - R) - \Gamma_{2j} \in \Sigma[x,\delta] \tag{11c}$$
$$\Gamma_{\#j} = s_{\#1} m_1 + ... s_{\#i} m_i + ... + s_{\#j} m_j, \quad \# = 0,1,2 \tag{11d}$$

Then the conditions of Theorem 1 are satisfied $\forall \delta \in \Delta$ and $\varepsilon_{R,\gamma} \subseteq \mathcal{R}_\delta$.

This Lemma compounds known results, previously commented in this article, and provides a novel recipe for the determination of robustly invariant sets. If for a moment the terms $\Gamma_{\#j}$ are ignored, the SOS constraints (11a-11b-11c) are an application of the P-Satz (Lemma 1) to enforce respectively the set containments (9a-9b-9c). The combined application of Lemma 1 and Lemma 2 allows also the inequalities defining the set Δ in (10) to be expressed as set containments, and then enforced as SOS conditions. This results in the terms $\Gamma_{\#j}$ defined in (11d), which guarantee that the conditions certifying that the level set is an ERA of the system (Theorem 1) are verified $\forall \delta \in \Delta$. That is, $\varepsilon_{R,\gamma}$ is an rERA of the origin.

The corresponding program to enlarge the provable rERA of a given system is:

Program 1.

$$\max_{s_1,s_2,s_{0i},s_{1i},s_{2i} \in \Sigma[x,\delta];\ s_0,V_N \in \mathbb{R}[x,\delta];\ R \in \mathbb{R}[x]} \gamma \quad \text{subject to conditions (11a)–(11b)–(11c)} \tag{12}$$

Note that V_N enters affinely in (11), whereas there are bilinear terms involving the multipliers s_i, γ and R. When the objective function is one of the two terms in the bilinearity (e.g. $s_0\gamma$), it was demonstrated in (Seiler and Balas, 2010) that the problem is quasiconvex and thus, the global optimum can be computed via cost bisection. However, the terms bilinear in s_i and R (e.g. $s_0 R$) make the above program non-convex. This can be handled with local BMI solvers (Kocvara and Stingl, 2006) or by means of iterative schemes. The latter approach is followed here and the following algorithm is proposed:

Algorithm 1.
Output: the level set $\varepsilon(R,\gamma)$.
Input: polynomials $R^0 \in \mathbb{R}[x]$, $V_N^0 \in \mathbb{R}[x,\delta]$ satisfying (11).

Step 1 : solve for $s_0, s_1, s_2, s_{0i}, s_{1i}, s_{2i}, \gamma$ by:

$$\max_{s_1,s_2,s_{0i},s_{1i},s_{2i} \in \Sigma[x,\delta];\ s_0,V_N \in \mathbb{R}[x,\delta]} \gamma$$
$$-\nabla R^0 f - s_0(\gamma - R^0) \quad - \Gamma_{0j} \in \Sigma[x,\delta]$$
$$V_N^0 - s_1(\gamma - R^0) - \Gamma_{1j} \qquad \in \Sigma[x,\delta]$$
$$-\nabla V_N^0 f - s_2(\gamma - R^0) - \Gamma_{2j} \in \Sigma[x,\delta]$$

Step 2 : solve for V_N, γ by:

$$\gamma_2 = \max_{V_N \in \mathbb{R}[x,\delta]} \gamma$$
$$-\nabla R^0 f - \bar{s}_0(\gamma - R^0) - \bar{\Gamma}_{0j} \in \Sigma[x,\delta]$$
$$V_N - \bar{s}_1(\gamma - R^0) - \bar{\Gamma}_{1j} \qquad \in \Sigma[x,\delta]$$
$$-\nabla V_N f - \bar{s}_2(\gamma - R^0) - \bar{\Gamma}_{2j} \in \Sigma[x,\delta]$$

Step 3 : solve for s_3, R, γ by:

$$\max_{s_3 \in \Sigma[x,\delta];\ R \in \mathbb{R}[x]} \gamma$$
$$-\nabla R f - \bar{s}_0(\gamma - R) - \bar{\Gamma}_{0j} \quad \in \Sigma[x,\delta]$$
$$\bar{V}_N - \bar{s}_1(\gamma - R) - \bar{\Gamma}_{1j} \qquad \in \Sigma[x,\delta]$$
$$-\nabla \bar{V}_N f - \bar{s}_2(\gamma - R) - \bar{\Gamma}_{2j} \in \Sigma[x,\delta]$$
$$(\gamma - R) - s_3(\gamma_2 - R^0) \qquad \in \Sigma[x,\delta]$$

with $\bar{\Gamma}_{\#j} = \bar{s}_{\#1} m_1 + ... \bar{s}_{\#i} m_i + ... + \bar{s}_{\#j} m_j$.
The superscript 0 indicates that the functions hold the value calculated at the end of the previous iteration (or their initializations, if at the first iteration), whereas the symbol *bar* is used for quantities optimised within the same iteration (at previous steps).
The scheme consists of one quasi-convex step (Step 1) and two convex steps (Steps 2-3). Each step has a specific task: Step 1 provides the multipliers for the next two steps; Step 2 calculates the function V_N; and Step 3 evaluates the sought level set $\varepsilon(R,\gamma)$ based on $\bar{V}_N$ from the previous step. Note also that the last SOS constraint in Step 3

is introduced to ensure that $\varepsilon(R^0,\gamma) \subseteq \varepsilon(R,\gamma)$, i.e. the solution (at Step 3) is a set that strictly contains the previous one (at Step 2).
The size of the ERA γ is maximised throughout each iteration, although Steps 2-3 can also be solved as simple feasibility problems. In this regard, note that the optimality of the solution is already prevented by the non-convexity of Program 1, and that the algorithm ensures in any case that the ERA is non-decreasing. Therefore, resorting to just feasibility when maximization fails is a viable solution.

3.2 Numerical aspects

Algorithm 1 requires initializations for R and V_N. A first option is to use any quadratic LF proving asymptotic stability of the nominal linearised system (provided that the associated Jacobian is Hurwitz), named here V_{lin}, which automatically satisfies (11) for the hypotheses discussed in Section 2. Alternatively, the corresponding functions obtained with the ERA calculation (e.g. using the algorithm for the nominal case in (Valmorbida and Anderson, 2017)) can be used.
The initialization is deemed an important aspect of the search for rERAs since the obtained local optimum is very sensitive to the initial guess. In this regard, it is worth stressing the importance of the fact that Algorithm 1 is initialized with both functions R and V_N. This feature can be favourable when preliminary estimations of the *shape* of the ERA (i.e. R) and a LF (i.e. V_N) are available in that the search can be seeded with them. When not specified, the algorithms are initialised here with V_{lin}.

Another interesting aspect is that the independent variables of the optimization include the states of the system x and the uncertain parameters δ. The polynomial multipliers s can thus potentially be function of both x and δ (as reported in Algorithm 1), but in practice there is a trade-off between computational time and accuracy.
In this regard, one of the advantages of this formulation is that the level set function is $R = R(x)$ (i.e. uncertain parameter-independent), whilst V_N is parameter dependent, i.e. $V_N(x,\delta)$. On the one hand, this is a less conservative approach than the one represented by parameter-independent LF level sets. On the other, the fact that $\varepsilon_{R,\gamma}$ is parameter-independent avoids the computation of the intersection of the parameterised estimates, resulting in a more accurate and easier to visualise outcome. This favorable twofold behaviour is the result of using two distinct functions, R and V_N, which allows for greater flexibility in the optimization. Interestingly, the need to optimize over two functions does not necessarily imply a rise in total run time when compared to the LF-based approaches, as discussed in Section 4.

The description of the set in (10) entails the definition of the polynomials m_i, which depend on the type of uncertainties featuring the system. This work will focus on parametric uncertainties, and thus possible definitions will be discussed for this case. Let us denote $\underline{\delta}_i$ and $\overline{\delta}_i$ respectively the minimum and maximum allowed values for each uncertain parameter δ_i. Then, at each parameter a polynomial m_i can be associated:

$$
\begin{aligned}
&m_i(\delta_i) = -(\delta_i - \underline{\delta}_i)(\delta_i - \overline{\delta}_i)\\
&\delta_i \in \Delta \Longleftrightarrow m_i(\delta_i) \geq 0\\
&\delta = [\delta_1 ... \delta_i ... \delta_j]^T
\end{aligned}
\tag{13}
$$

Recalling the definition of $\Gamma_{\#j}$ in (11d), it is worth noting that for each employed m_i there are three multipliers s_{0i}, s_{1i}, s_{2i} (one for each constraint). Therefore, as the number of uncertain parameters increases, so does the size of the associated optimization problem. However, an alternative solution is to define a single polynomial m_c:

$$
\begin{aligned}
&m_c(\delta) = -\sum_{i=1}^{j}(\delta_i - \underline{\delta}_i)(\delta_i - \overline{\delta}_i) = \sum_{i=1}^{j} m_i(\delta_i)\\
&\delta \in \Delta \Longrightarrow m_c(\delta) \geq 0
\end{aligned}
\tag{14}
$$

which specializes (11d) to $\Gamma_{\#1} = s_{\#c} m_c$.
This definition gives only a sufficient condition (as opposed to the one in (13) which is also necessary), because there are values of $\delta \notin \Delta$ for which the inequality $m_c(\delta) \geq 0$ is satisfied. Therefore, the obtained rERA is valid for a larger range of uncertainties. However, the adoption of m_c has the advantage of adding only 3 multipliers s_{0c}, s_{1c}, s_{2c}, and therefore a trade-off between computational speed versus over-conservatism arises.

4. RESULTS

In this section, the capability of the framework built in Section 3 to study rERA based on an invariant set formulation are applied to two nonlinear (polynomial) study cases. All the analyses are performed on a 3.6 GHz desktop PC with 16 GB RAM.

4.1 Van der Pol oscillator

The Van der Pol (VdP) oscillator is a nonlinear system with 2 states. In (Topcu and Packard, 2009) the problem with an uncertain scalar parameter $\delta_1 \in [-1, 1]$ was considered:

$$
\begin{aligned}
\dot{x}_1 &= -x_2(1 + 0.2\delta_1)\\
\dot{x}_2 &= x_1 + (x_1^2 - 1)x_2
\end{aligned}
\tag{15}
$$

The VdP steady-state solutions are characterized, for all the values of δ_1 within the considered range, by an unstable limit cycle and a stable origin. The ROA for this system is the region enclosed by its limit cycle and thus can be easily obtained from the numerical solution of the associated ordinary differential equations. Its estimation was performed in (Topcu and Packard, 2009) via parameters-independent LF level sets enforcing the SOS constraints on both vertices of the uncertainty range (in full generality, on each vertex of the uncertain polytope).
Fig. 1 shows the rERA given by Algorithm 1 (curve *IS*), along with the predictions obtained with an in-house implementation of the algorithm from (Topcu and Packard, 2009) (curve *LF*, in good agreement with the results displayed therein), and the unstable limit cycles of the system corresponding to eight values of δ_1 across its range. The algorithm is initialised with the functions V_N and R from nominal analyses. The cases with $\partial(V_N) = \partial(R) = 4$ is considered here. Note that $V_N(x, \delta_1)$ is built from monomials in x and δ_1 up to degree $\partial(V_N)$, with the property that $V_N(0, \cdot) = 0$ as required by Lemma 3.
Observe that the rERA obtained with the algorithm based

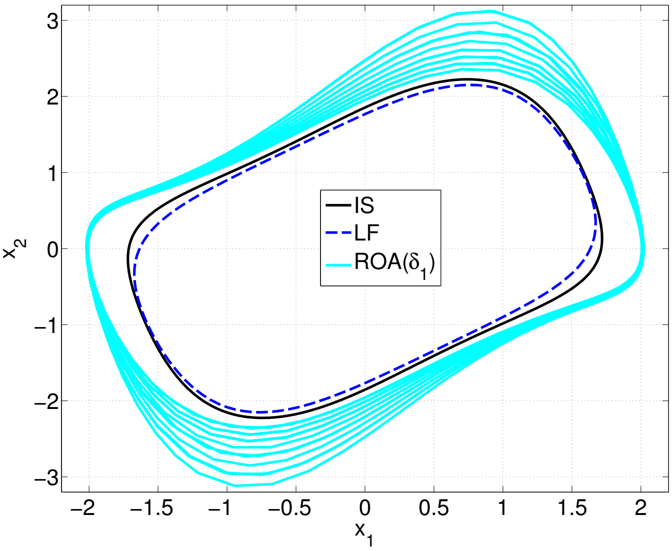

Fig. 1. ERA of VdP ($\partial(V_N, R) = 4$).

on invariant sets is larger than the level set from (Topcu and Packard, 2009). Note also that the curve *IS* is close, along much of its perimeter, to the smallest of the curves corresponding to the ROA of the system (providing an upper bound on the guaranteed safe region). When the degree of the optimised functions is increased (e.g. $\partial(V_N) = \partial(R) = 6$), the rERA expands in the directions where the gap with $ROA(\delta_1)$ is larger, resulting on the other hand in longer simulations. In this regard, the total run-time to obtain the curve *IS* in Fig. 1 was 297 *seconds*, as opposed to 175 *seconds* for the curve *LF*.

4.2 Controlled short-period aircraft dynamics

The second test case consists of a closed-loop nonlinear short-period (SP) model of the longitudinal dynamics of an aircraft. It features 3 open-loop states (pitch rate z_1, angle of attack z_2, pitch angle z_3) and 2 controller states η_1, η_2. In (Topcu et al., 2008; Topcu and Packard, 2009), the case where two parametric uncertainties δ_1 and δ_2 affect the open loop dynamics was studied:

$$
\begin{aligned}
\dot{z} &= \begin{bmatrix} -3 & -1.35 & -0.56 \\ -0.91 & -0.64 & -0.02 \\ 1 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1.35 - 0.04 z_2 \\ 0.4 \\ 1 \end{bmatrix} u \\
&+ \begin{bmatrix} (1+\delta_1)(0.08 z_1 z_2 + 0.44 z_2^2 + 0.01 z_2 z_3 + 0.22 z_2^3) \\ (1+\delta_2)(-0.05 z_2^2 + 0.11 z_2 z_3 - 0.05 z_3^2) \\ 0 \end{bmatrix} \\
\dot{\eta} &= \begin{bmatrix} -0.6 & 0.09 \\ 0 & 0 \end{bmatrix} \eta + \begin{bmatrix} -0.06 & -0.02 \\ -0.75 & -0.28 \end{bmatrix} y \\
y &= [z_1 \quad z_3]^T; \quad u = \eta_1 + 2.2 \eta_2; \\
\delta_1&, \delta_2 \in [-0.1, 0.1]
\end{aligned} \tag{16}
$$

By defining $x = [z \quad \eta]^T$ and $\delta = [\delta_1 \quad \delta_2]^T$, the system is recast in the class of nonlinear dynamics given by (7).

In the aforementioned references the adopted algorithms were based on parameter-independent LF level sets. Namely, (Topcu and Packard, 2009) devised a suboptimal strategy to avoid enforcing the SOS constraints at each vertex of the polytope (computationally demanding already for this size of problems); and in (Topcu et al., 2008) a branch-and-bound refinement of the suboptimal algorithm consisting in partitioning the uncertainty set and determining a different parameter-independent LF for each cell was employed. In both cases, the algorithms provided the rERA only in the form of $\varepsilon_{p,\beta}$, where $p(x) = x^T N x$ and $N \in \mathbb{R}^{n\times n}$, $N = N^T > 0$ is a *given* matrix defining the shape of the ellipsoid to which the search of inner estimates of the ROA is restricted. This is because a unique Lyapunov function V certifying the ROA over the entire uncertainty set could not be computed.

Note that now the system has more than 2 states, therefore *projections* of the ERA onto particular planes are employed to graphically visualize the predictions. In general, the analyst will focus on the states which are supposed to experience larger perturbations during the operation of the system. In this work, the z_1-z_2 phase-plane (Fig. 2) will be displayed since the studied nonlinearities arise from their dynamics. Further, to provide as much information relative to the analyses as possible, Fig. 3 shows the projections onto the z_1-z_3 plane. The same nomenclature as in the previous plot applies, and the degree of the optimized polynomials V_N and R is 2.
The largest estimate available in the published literature, taken from (Topcu et al., 2008) and obtained with quartic LFs employing the suboptimal (branch-and-bound refined) algorithm, corresponds to $\beta = 11.1$ and $p = x^T x$ and is reported in here for comparison (*LF* in the figure). Note also that the analyses displayed in the figures below are obtained describing the uncertainty set with a single polynomial $m_c(\delta_1, \delta_2)$ following the definition in (14).

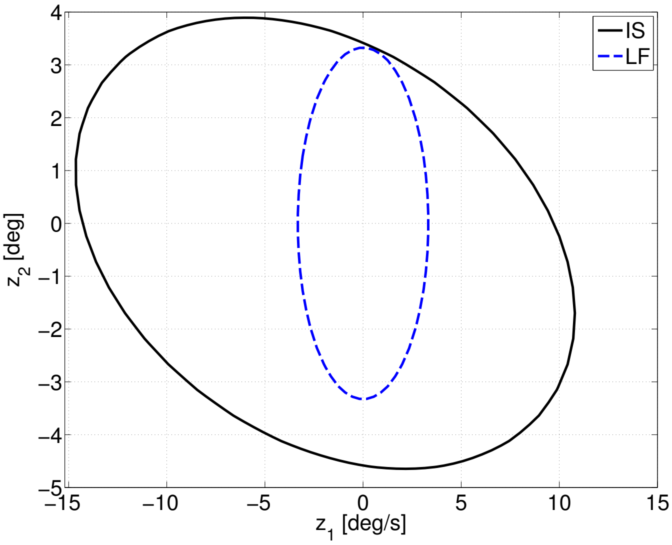

Fig. 2. Projection of the rERA of SP onto z_1-z_2 plane.

The results showcase that the proposed invariant sets-based algorithm markedly outperforms the estimation given with the LF level set approach (which was obtained with a higher degree for the LFs). This aspect can be ascribed primarily to the less conservative formulation of the problem (invariant set, see Lemma 3, versus LF level sets), but also to the flexibility given by Algorithm 1 in optimizing the *shape* of the level set throughout the iterations. In some of the LF-based algorithms proposed to handle the case with uncertainties, as in (Topcu et al., 2008), the shape of the rERA (p) is fixed a priori and the

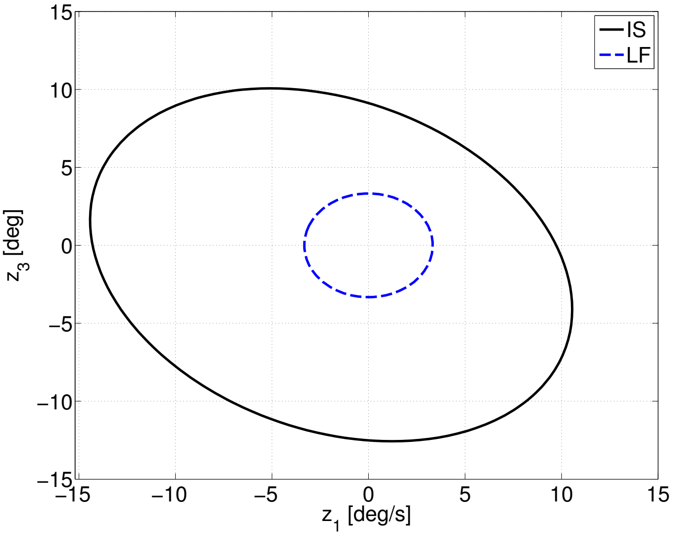

Fig. 3. Projection of the rERA of SP onto z_1-z_3 plane.

only optimized quantity is the size β such that the largest possible set ($\varepsilon_{p,\beta}$) verifying the conditions defining the rERA is achieved. This does not exploit the directionality of the ROA, and is exemplified in both Figs. 2-3 where the curves *IS* are ellipses with distinct axes, whereas the curves *LF* are circles. A possible improvement on this aspect consists in using the value of p from a previous rERA calculation to initialize the analyses.

The total run-time to obtain the invariant set rERA was 1890 *seconds*. While in (Topcu et al., 2008) there is no reference to computational time or size of the problem, in (Topcu and Packard, 2009) a smaller estimation (i.e. without branch-and-bound refinement) was achieved in approximately 2300 *seconds*, and this can be taken as a lower bound on the total processing time of algorithm *LF*. Note that the SP system features 5 states, nonlinearities up to degree 3, and 2 uncertainties, and it represents a challenging test case from the computational point of view (Topcu and Packard, 2009). Indeed, it is well-known that SOS suffer of the so-called *scalability*, i.e. the significant growth in simulation time as the size of the analysed system increases. The research community is working on solutions to tackle this well recognised issue (see for example (Ahmadi et al., 2017) where a survey on recent advances is presented) with the aim to provide efficient numerical tools that can make SOS-based programs (as the one presented in this article) more amenable for high order systems.

5. CONCLUSION

This article considers the problem of estimating the region of attraction of systems described by polynomial vector fields and subject to modelling uncertainties. A recently proposed formulation based on invariant level sets is adopted as the theoretical foundation to propose a recipe for the computation of robust inner estimates.

The rERA is characterized with set containments which are enforced as SOS constraints using known results from real algebraic geometry. The ensuing optimization problem is bilinear and thus an iterative scheme is proposed to enlarge provable regions. Features of the algorithms are commented, and possible solutions to lower the computational cost due to the inclusion of the uncertainties in the problem are discussed. It is emphasized the advantage, compared to other established algorithms, resulting from optimising two functions, one directly defining the level set (R), which is parameter-independent, and the other an auxiliary one (V_N), which is parameter-dependent. This approach can hence in principle retain the advantages of these two typically conflicting aspects.

Preliminary results show that the proposed algorithm leads to larger estimations of the rROA than the LF level-set approaches considered for reference. As the size of the analysed system is increased, this approach seems also to be more efficient in that it features a smaller run-time.

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